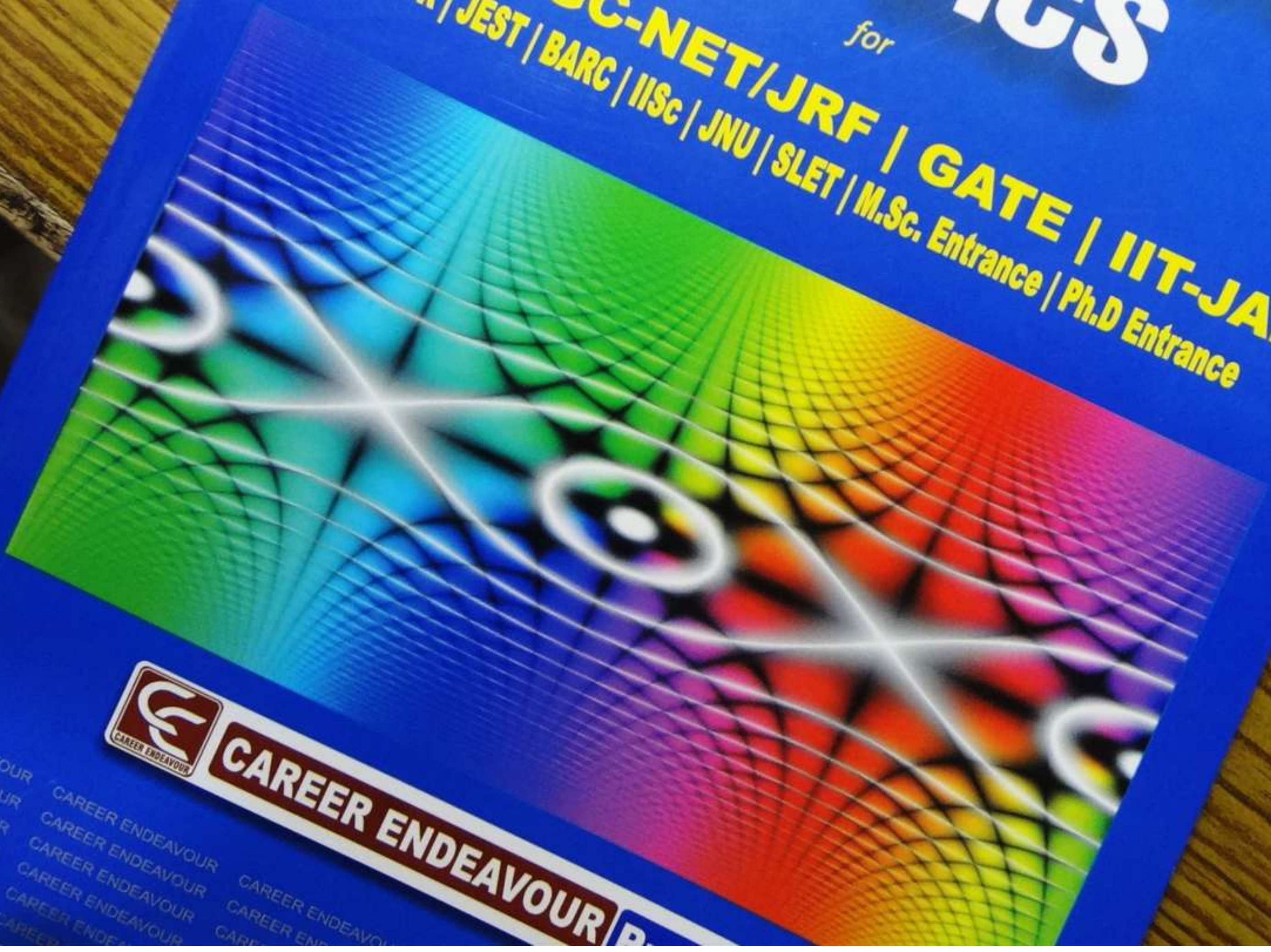
MATHEMATICAL PHYSICS

2e

for

CSIR-UGC-NET/JRF | GATE | IIT-JAM

TIFR | JEST | BARC | IISc | JNU | SLET | M.Sc. Entrance | Ph.D Entrance



CAREER ENDEAVOUR

Preface to the Second Edition

Mathematical Physics form an important part of the syllabus for the graduate and post-graduate physics courses of various Indian Universities. While teaching this subject, particularly in physics honours classes, we very often refer to various standard books for the various topics of Mathematical Physics. But none of these books is fully sufficient for the preparation of different competitive examinations at national and state levels such as CSIR-UGC-NET/JRF, IIT-JAM, GATE, TIFR, JEST, BARC etc. In all these competitive examinations, the section of Mathematical Physics carries a high percentage of weightage of the total marks as follows:

- In every year CSIR-UGC-NET paper contains questions of at least 35 to 40 marks from this section out 200 marks.
- In every year GATE paper contains questions of at least 10 to 12 marks from this section out 100 marks.
- In every year IIT-JAM paper contains questions of at least 10 to 12 marks from this section out 100 marks.

The purpose of the present book is to provide a brief discussion of every important topic of the Mathematical Physics syllabus. Each chapter contains a significant number of solved problems some of which contain subtle points, not explicit in the text and hence will help the students to attain a deeper understanding of the ideas presented in the book. Moreover, the problems that are asked in CSIR-UGC-NET, GATE and IIT-JAM examinations till date, has been solved at the end of each chapter. A large variety of problems given with answers at the end of each chapter for the practice of the students, will surely help to grow self-confidence.

It is hoped that the book will be able to meet up all the requirements of the students preparing for the various competitive examinations. However, any constructive criticism and suggestions for the improvement of the book will always be highly appreciated. Much care has been taken to minimize the typographical errors, yet if any noticed, then your suggestions are invited with a view to improve the book in future editions.

I would like to thank my parents, family members, students, friends and specially Anwesha Chakraborty who have given constant support in the development of the book. Thanks are due to the unit of *Career Endeavour Publications* for making sincere efforts towards timely publication of the book. Thanks to Mr. Surajit Kuila for his constructive support in doing corrections and proofreading. Last but not the list, a very special thanks to Mohammad Firoz for typing, formatting and designing of the book.

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(Author)

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Chapter 1

VECTOR ALGEBRA

1.1 Basic Review of Vectors

Definition:

Physical quantities having both magnitude and a definite direction in space. It should follow the law of vector addition.

Example: Velocity, Acceleration, Momentum, Force, Electric Field, Torque, etc.

Note: Current is a physical quantity that has both magnitude and direction but it does not follow the law of vector addition. So, current is a scallar quantity.

Various type of vectors:

(1) Equal vectors: Vectors having same magnitude and same direction.

(2) Null Vectors: Vectors having coincident initial and terminal point i.e. its magnitude is zero and it has any arbitrary direction.

(3) Unit Vector: Vector having unit magnitude. Unit vector along $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

(4) Reciprocal Vector: Vector having same direction as $\vec{a}$ but magnitude reciprocal to that of $\vec{a}$, is known as the reciprocal vector of $\vec{a}$. Reciprocal vector of $\vec{a}$ is $\vec{a}^{-1} = \frac{1}{|a|}\hat{a}$

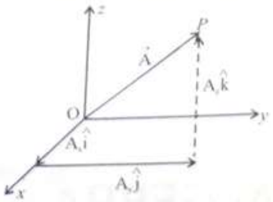
(5) Negative Vector: Vectors having same magnitude as $\vec{a}$ but direction opposite to that of $\vec{a}$, is known as the negative vector of $\vec{a}$. Negative vector as $\vec{a}$ is $-\vec{a} = -|a|\hat{a}$

Orthogonal Resolution of Vectors:

Any vector $\vec{A}$ in the 3-D right- handed rectangular cartesian coordinate system can be represented as $\overrightarrow{OP} = \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in direction of x, y and z axis respectively and A_x, A_y, A_z are the rectangular components of vector $\vec{A}$ along x, y, z axis.

Magnitude of vector $\vec{A}$ is $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector along $\vec{A}$ is $\hat{A} = \vec{A} / |\vec{A}| = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right) / \sqrt{A_x^2 + A_y^2 + A_z^2}$



Direction cosines of vector $\vec{A}$:

If $\vec{A}$ makes angles α, β, γ with x, y and z axes respectively, then direction cosines of $\vec{A}$ are defined as

$$l = \cos\alpha = \frac{A_x}{A}; \ m = \cos\beta = \frac{A_y}{A}; \ n = \cos\gamma = \frac{A_z}{A} \quad \text{and} \quad l^2 + m^2 + n^2 = 1$$

Note: Unit vector along $\vec{A}$ can be written as $\hat{A} = l\hat{i} + m\hat{j} + n\hat{k}$

1.2 Products of Vectors

Scalar Product or Dot Product:

Dot product of two vectors $\vec{a}$ and $\vec{b}$ are defined as

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Note:

(i) Dot product is commutative in nature i.e. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(ii) For two mutually perpendicular vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0$

(iii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot i = 0$, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(iv) If $\vec{a} = a_z\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}.\vec{b} = a_xb_x + a_yb_y + a_zb_z$

(v) Projection of $\vec{A}$ on $\vec{B} = \vec{A}.\hat{B}$

Vector Product or Cross Product :

Cross product of two vectors $\vec{a}$ and $\vec{b}$ are defined as

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note:

(i) Cross product is not commutative i.e. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

(ii) For two collinear vectors (parallel or anti-parallel vectors) $\vec{a}\times\vec{b} = 0$.

(iii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$, $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

(iv) If $\vec{a} = a_z\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

Example 1: The range of x for which between the vectors $\vec{A} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{B} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse, is equal to

(a) $x < 0$ (b) $x > 1/2$ (c) $0 < x < 1/2$ (d) None of these

Soln: Since, $90^0 < \theta < 180^0$, then $\cos\theta < 0 \Rightarrow \vec{A}.\vec{B} < 0 \Rightarrow 14x^2 - 7x < 0 \Rightarrow x(2x-1) < 0$

Case I: $x > 0$ and $(2x-1) < 0 \Rightarrow x < \frac{1}{2}$. Therefore, $0 < x < \frac{1}{2}$

Case II: $x < 0$ and $(2x-1) > 0 \Rightarrow x > \frac{1}{2}$. Both of them cannot be satisfied simultaneously.

Correct option is (c)

Example 2: Given: $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ are the sides of

(a) an isosceles triangle (b) a right-angled triangle

(c) an equilateral triangle (d) none of these

Soln: For the given set of vectors, $\vec{a}.\vec{c} = 0 \Rightarrow \vec{a}$ is perpendicular to $\vec{c}$ and

$$|\vec{a}|^2 + |\vec{c}|^2 = 14 + 21 = 35 = |\vec{b}|^2$$

Therefore, $\vec{a}$, $\vec{c}$ will correspond to base and height respectively and $\vec{b}$ will correspond to hypotenus i.e. the given set of vectors are sides of a right-angled triangle.

Correct option is (b)

Example 3: Given : $\vec{P} = 2\hat{i} - 3\hat{j}, \vec{Q} = \hat{i} + \hat{j} - \hat{k}, \vec{R} = 3\hat{i} - \hat{k}$, the vector $\vec{V}$, which is orthogonal to $\vec{P}$ and $\vec{Q}$ and having unit scalar product with $\vec{R}$, can be estimated to be

(a) $(3\hat{i} + 2\hat{j} + 5\hat{k})/4$ (b) $(3\hat{i} + 2\hat{j} - 5\hat{k})/4$ (c) $(3\hat{i} - 2\hat{j} + 5\hat{k})/4$ (d) $(3\hat{i} - 2\hat{j} - 5\hat{k})/4$

Soln: Unit vector along $\vec{V}$ is $\hat{V} = \frac{\vec{P}\times\vec{Q}}{|\vec{P}\times\vec{Q}|} = \frac{3\hat{i} + 2\hat{j} + 5\hat{k}}{\sqrt{38}}$

According to the question, $\vec{V}.\vec{R} = 1 \Rightarrow (V\hat{V}).\vec{R} = 1 \Rightarrow V = \frac{\sqrt{38}}{4}$

Correct option is (a)

Scalar Triple Product:

Scalar triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=[abc]$$

Note:

(i) $\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$ i.e. $[abc]=[bca]=[cab]$

(ii) Volume of a parallelopiped having $\vec{a},\vec{b},\vec{c}$ as concurrent edges is :V=$|\vec{a}\cdot(\vec{b}\times\vec{c})|$

(iii) If $\vec{a},\vec{b},\vec{c}$ are coplanar vectors, then, $\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})=0$

Vector Triple Product:

Vector triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ are defined as

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$$

It represents a vector coplanar with $\vec{b}$ and $\vec{c}$ and perpendicular to $\vec{a}$

Note:

(i) $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=0$

Example 4: The value of 'm' for which $\vec{A}=\hat{i}-\hat{j}-2\hat{k},\vec{B}=3\hat{i}+5\hat{j}+6\hat{k},\vec{C}=-\hat{i}+4\hat{j}+m\hat{k}$ will be co-planar is

(a) 13/2 (b) 7/2 (c) 5/2 (d) 11/2

Soln: Since, $\vec{A},\vec{B},\vec{C}$ are co-planar, $\vec{A}.\left(\vec{B}\times\vec{C}\right)=0\Rightarrow\begin{vmatrix} 1 & -1 & -2 \\ 3 & 5 & 6 \\ -1 & 4 & m \end{vmatrix}=0\Rightarrow m=\dfrac{13}{2}$

Example 5: If $\vec{b}=\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)$, then $\vec{b}$ can be simplified to

(a) 0 (b) $\vec{a}$ (c) $2\vec{a}$ (d) None of these

Soln: $\vec{b}=\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)$

$=\vec{a}\left(\hat{i}\hat{i}\right)-\hat{i}\left(\hat{i}.\vec{a}\right)+\vec{a}\left(\hat{j}.\hat{j}\right)-\hat{j}\left(\hat{j}.\vec{a}\right)+\vec{a}\left(\hat{k}.\hat{k}\right)-\hat{k}\left(\hat{k}.\vec{a}\right)=3\vec{a}-\vec{a}=2\vec{a}$

Example 6: Three unit vectors $\vec{a},\vec{b},\vec{c}$ ($\vec{b}$ and $\vec{c}$ are not parallel) are such that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\dfrac{\sqrt{3}}{2}\vec{c}$. The angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, respectively are

(a) $30^0,90^0$ (b) $150^0,90^0$ (c) $60^0,90^0$ (d) $90^0,30^0$

Soln: $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\dfrac{\sqrt{3}}{2}\vec{c}\Rightarrow\vec{b}\left(\vec{a}.\vec{c}\right)-\vec{c}\left(\vec{a}.\vec{b}\right)=\dfrac{\sqrt{3}}{2}\vec{c}\Rightarrow\vec{a}.\vec{c}=0$ and $\vec{a}.\vec{b}=-\dfrac{\sqrt{3}}{2}$

The angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are $150^0,90^0$ respectively.

1.3 Gradient, Divergence and Curl

Gradient of a Scalar Field:

Gradient of a continuously differentiable scalar function $\phi(x, y, z)$ is mathematically defined as:

$$\text{grad}\phi=\vec{\nabla}\phi=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

where, $\vec{\nabla}=\dfrac{\partial}{\partial x}\hat{i}+\dfrac{\partial}{\partial y}\hat{j}+\dfrac{\partial}{\partial z}\hat{k}\Rightarrow$ 'del' or 'grad' or 'nabla' operator

Physical interpretation: Gradient of scalar function ϕ at any point P(x, y, z) is a vector quantity, whose magnitude is equal to the rate of change of scalar function ϕ with distance along the normal to level surface and its direction is along the normal to the level surface at that point. The direction of $\vec{\nabla}\phi$ is that along which the rate of change of ϕ is maximum.

Level surface: At each point of the level surafce, the value of scalar function ϕ will be same.

Example: Equipotential surface.

Note:

(i) Normal vector to the level surface: $\vec{\nabla}\phi$

(ii) Unit normal vector to the level surface: $\hat{n}=\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

Directional Derivative: Directional derivative of ϕ in the direction of $\vec{A}$ is defined as rate of change of ϕ with distance along the direction of $\vec{A}$. It is mathematically defined as the component of $\vec{\nabla}\phi$ in the direction of vector $\vec{A}$ i.e. $\vec{\nabla}\phi\cdot\hat{A}=\vec{\nabla}\phi\cdot\dfrac{\vec{A}}{|\vec{A}|}$

Tangent Plane and Normal to the level surface:

Consider $\phi(x,y,z)=c$ be the equation of a level surface. and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ be the position vector of any point P (x,y,z) on this surface.

Tangent plane at P: $\vec{\nabla}\phi$ is a vector normal to the surface i.e. it is perpendicular to the tangent plane at P. Let, $\vec{R}=X\hat{i}+Y\hat{j}+Z\hat{k}$ be the position vector of any point on the tangent plane at P to the surface. Therefore, $\vec{R}-\vec{r}=(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}$ lies in the tangent plane at P and it will be perpendicular to $\vec{\nabla}\phi$ i.e. $\left(\vec{R}-\vec{r}\right).\vec{\nabla}\phi=0$

$\Rightarrow (X-x)\frac{\partial\phi}{\partial x}+(Y-y)\frac{\partial\phi}{\partial y}+(Z-z)\frac{\partial\phi}{\partial z}=0$, which is the tangent plane at point P.

Normal at P: Let, $\vec{R}=X\hat{i}+Y\hat{j}+Z\hat{k}$ be the position vector of any point on the normal at P to the surface. Therefore, $\vec{R}-\vec{r}=(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}$ lies along the normal at P and it will be parellel to $\vec{\nabla}\phi$ i.e. $\left(\vec{R}-\vec{r}\right)\times\vec{\nabla}\phi=0$, which is the vector equation of the normal at point P to the given surface.

Divergence of a vector field:

Divergence of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}.\vec{V}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=\text{Scalar quantity}$$

Physical Interpetation: Divergence of $\vec{V}$ at point P(x, y, z) is defined as the outward flux of the vector field $\vec{V}$ per unit volume enclosed by an infinitesimal closed surface surrounding point 'P'.

Note:

(i) If $\vec{\nabla}.\vec{V}=0$, then $\vec{V}$ is known as solenoidal vector field.

(ii) If $\vec{\nabla}.\vec{V}=$ negative, then $\vec{V}$ is known as sink field i.e. vector lines are going inward.

(iii) If $\vec{\nabla}.\vec{V}=$ positive, then $\vec{V}$ is known as source field i.e. vector lines are the going outward.

(iv) $\vec{\nabla}\cdot\left(\vec{U}+\vec{V}\right)=\vec{\nabla}\cdot\vec{U}+\vec{\nabla}\cdot\vec{V}$

(v) $\vec{\nabla}.\left(u\vec{V}\right)=(\vec{\nabla}u)\cdot\vec{V}+u(\vec{\nabla}.\vec{V})$

Curl of a vector field:

Curl of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla}\times\vec{V}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(v_x\hat{i}+v_y\hat{j}+v_z\hat{k}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}$$

It is the measurement of rotation of vector field and the direction of curl of vector is along axis of rotation.

Note:

(i) If $\vec{\nabla}\times\vec{V}=0$, then $\vec{V}$ is known as an irrotational vector and we can write $\vec{V}=\vec{\nabla}\phi$

(ii) If $\vec{\nabla}\times\vec{V}\neq 0$, then $\vec{V}$ is known as rotational vector.

(iii) $\vec{\nabla}\times(\vec{U}+\vec{V})=\vec{\nabla}\times\vec{U}+\vec{\nabla}\times\vec{V}$

(iv) $\vec{\nabla}\times(u\vec{V})=u\vec{\nabla}\times\vec{V}+(\vec{\nabla}u)\times\vec{V}$

Important Vector Identities:

If ϕ, ψ are scalar point functions and $\vec{A}$, $\vec{B}$ are vector point functions in certain region then.

(1) $\vec{\nabla}(\phi+\psi)=\vec{\nabla}\phi+\vec{\nabla}\psi$

(2) $\vec{\nabla}(\phi\psi)=\phi\vec{\nabla}\psi+\psi\vec{\nabla}\phi$

(3) $\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$

(4) $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

(5) $\vec{\nabla}(\vec{A}\cdot\vec{B})=\vec{A}\times(\vec{\nabla}\times\vec{B})+(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{A}$

(6) $\vec{\nabla}\cdot(\phi\vec{A})=\phi(\vec{\nabla}\cdot\vec{A})+\vec{A}\cdot\vec{\nabla}\phi$

(7) $\vec{\nabla}\times(\phi\vec{A})=\phi(\vec{\nabla}\times\vec{A})+(\vec{\nabla}\phi)\times\vec{A}$

(8) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$

(9) $\vec{\nabla}\times(\vec{\nabla}\phi)=0$

(10) $\vec{\nabla}\times(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$

(11) $\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})$

(12) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$

(13) For a constant vector $\vec{A}$, $\vec{\nabla}\left(\vec{A}.\vec{r}\right)=\vec{A}$ and $\vec{\nabla}\times\left(\vec{A}\times\vec{r}\right)=2\vec{A}$

Example 7: Find the unit vector perpendicular to the surface $x^2+y^2+z^2=6$ at point P (2,1,1).

Soln: Unit normal to the surface $\hat{n}=\left.\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}\right|_{(2,1,1)}=\left.\frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}\right|_{(2,1,1)}=\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$

Example 8: Find the angle between the two surfaces $5x + y + z = 1$ and $3x + 3y + 3z = 5$.

Soln: Given, $\phi_1 : 5x + y + z = 1$ and $\phi_2 : 3x + 3y + 3z = 5$

Therefore, $\cos\theta = \frac{\vec{\nabla}\phi_1 . \vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1||\vec{\nabla}\phi_2|} = \frac{(5\hat{i} + \hat{j} + \hat{k}).(3\hat{i} + 3\hat{j} + 3\hat{k})}{\sqrt{27}.\sqrt{27}} = \frac{21}{27} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{9}\right)$

Example 9: Find the directional derivative of $f(x,y) = x^2y^3 - y^4$ at (2,1) in the direction given by the angle $\theta = \frac{\pi}{4}$.

Soln: Unit vector along which the directioanl derivative is to be calculated is

$$\hat{A} = \cos\frac{\pi}{4}\hat{i} + \sin\frac{\pi}{4}\hat{j} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$$

Therefore, directional derivative will be

$$\vec{\nabla}f\Big|_{(2,1)}.\hat{A} = \left[(2xy^3)\hat{i} + (3x^2y^2 - 4y^3)\hat{j}\right]_{(2,1)}.\frac{1}{\sqrt{2}}(\hat{i} + \hat{j}) = 6\sqrt{2}$$

Example 10: Show that, $\nabla^2 r^n = n(n+1)r^{n-2}$.

Soln: $\nabla^2 r^n = \vec{\nabla}.(\vec{\nabla}r^n) = \vec{\nabla}.(nr^{n-1}\hat{r}) = \vec{\nabla}.(nr^{n-2}\vec{r}) = \vec{\nabla}(nr^{n-2}).\vec{r} + nr^{n-2}(\vec{\nabla}.\vec{r})$

$= n(n-2)r^{n-3}\hat{r}.\vec{r} + 3nr^{n-2} = n(n-2)r^{n-2} + 3nr^{n-2} = n(n+1)r^{n-2}$

Example 11: The vector $r^n\vec{r}$ will solenoidal for $(r \neq 0)$

(a) n = 3
(b) n = -3
(c) n = 2
(d) n = -2

Soln: $\vec{\nabla}.(r^n\vec{r}) = \vec{\nabla}r^n.\vec{r} + r^n(\vec{\nabla}.\vec{r}) = nr^{n-1}\hat{r}.\vec{r} + 3r^n = (n+3)r^n$;

So the given vector will be solenoidal for n = - 3

Correct option is (b)

Example 12: If grad $\phi = \frac{\hat{r}}{r^4}$ & $\phi(r) = 0$ at $r = 1$, then find ϕ

Soln: $\vec{\nabla}\phi = \frac{\hat{r}}{r^4} \Rightarrow \frac{\partial\phi}{\partial r}\hat{r} = \frac{\hat{r}}{r^4} \Rightarrow \frac{\partial\phi}{\partial r} = \frac{1}{r^4} \Rightarrow \phi = -\frac{1}{3r^3} + C$

Since, at $r = 1$, $\phi(r) = 0 \Rightarrow 0 = -\frac{1}{3} + C \Rightarrow C = \frac{1}{3}$

Putting this we get $\phi(r) = -\frac{1}{3r^3} + \frac{1}{3}$

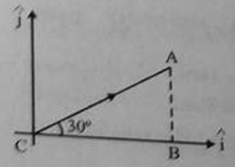
Example 13: For the function $\phi(x,y) = \frac{x}{x^2 + y^2}$, find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at (0, 2)

Soln: $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{x}{x^2 + y^2} = \hat{i}\left(\frac{1}{x^2 + y^2} - \frac{x(2x)}{(x^2 + y^2)^2}\right) - \hat{j}\frac{x(2y)}{(x^2 + y^2)^2}$

$= \hat{i}\frac{y^2 - x^2}{(x^2 + y^2)^2} - \hat{j}\frac{2xy}{(x^2 + y^2)^2}$

$\vec{\nabla}\phi$ at the point (0, 2)

$= \hat{i}\frac{4 - 0}{(0+4)^2} - \hat{j}\frac{2(0)(2)}{(0+4)^2} = \frac{\hat{i}}{4}$



Directional derivative at the point (0, 2) in the direction $\overline{CA}$ i.e. $(\cos 30^0\hat{i} + \sin 30^0\hat{j}) = \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$

$= \frac{\hat{i}}{4}.\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) = \frac{\sqrt{3}}{8}$

Example 14: Find the directional derivative of $\vec{V}^2$, where $\vec{V} = xy^2 + zy^2\hat{j} + xz^2\hat{k}$, at the point (2, 0, 3) in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).

Soln: $V^2 = \vec{V}.\vec{V} = V^2 = \vec{V}.\vec{V} = (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}).(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}) = x^2y^4 + z^2y^4 + x^2z^4$

$\vec{\nabla}V^2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$

$= (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4y^3z^2)\hat{j} + (2y^4z + 4x^2z^3)\hat{k}$

$\vec{\nabla}V^2$ at (2, 0, 3) $= (0 + 2\times2\times81)\hat{i} + (0+0)\hat{j} + (0 + 4\times4\times27)\hat{k}$

$= 324\hat{i} + 432\hat{k} = 108(3\hat{i} + 4\hat{k})$

Normal to $x^2 + y^2 + z^2 - 14 = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 14)$

$= (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

Normal vector at (3, 2, 1) $= 6\hat{i} + 4\hat{j} + 2\hat{k}$

Unit normal vector $= \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36+16+4}} = \frac{2(3\hat{i} + 2\hat{j} + \hat{k})}{2\sqrt{14}} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$

Direction derivative along the normal $= 108(3\hat{i} + 4\hat{k}).\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} = \frac{108\times(9+4)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}$

Example 15: A fluid motion is given by

$$\vec{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$$

is the motion irrotational? If so, find the velocity potential

Soln: Curl $\vec{v} = \vec{\nabla} \times \vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= (x\cos z + 2y - x\cos z - 2y)\hat{i} - [y\cos z - y\cos z]\hat{j} + (\sin z - \sin z)\hat{k} = 0$$

Hence, the motion is irrotational.

So, $\bar{v} = \vec{\nabla}\phi$ where ϕ is called velocity potential.

$$\Rightarrow \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = \bar{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Comparing both sides we get,

$$\frac{\partial\phi}{\partial x} = (y\sin z - \sin x) \Rightarrow \phi = xy\sin z + \cos x + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = (x\sin z + 2yz) \Rightarrow \phi = xy\sin z + y^2 z + f_1(x,z)$$

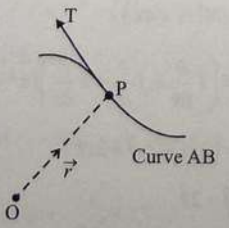
$$\frac{\partial\phi}{\partial z} = (xy\cos z + y^2) \Rightarrow \phi = xy\sin z + y^2 z + f_1(x,y)$$

Therefore, the velocity potential will be $\phi = xy\sin z + \cos x + y^2 z + c$

1.4 Line Integration of Vectors

The integration of a vector along a curve is known as line integration of vectors.

Let $\vec{F}(x,y,z)$ be a vector function and a curve AB.



Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

Component of $\vec{F}$ along a tangent PT at P = Dot product of $\vec{F}$ and unit vector along PT

$$= \vec{F}\cdot\frac{d\vec{r}}{ds}\left(\frac{d\vec{r}}{ds} \text{ is a unit vector along tangent PT}\right)$$

Line integral of $\vec{F}$ from A to B along the curve will be $= \int_c \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right)ds = \int_c \vec{F}\cdot d\vec{r}$

Note:

(i) If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{r}$$

(ii) If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V}\cdot d\vec{r}$ is called the circulation of $\vec{V}$ round closed curve c.

(iii) When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

(iv) Work done by a conservative field $\vec{A}$ in moving a particle from point P to Q will be

$$\int_P^Q \vec{F}\cdot\overrightarrow{dr} = \int_P^Q \vec{\nabla}\phi\cdot\overrightarrow{dr} = \int_P^Q d\phi = \phi_Q - \phi_P = \text{independent of path.}$$

(v) Work done by a conservative field $\vec{A}$ in moving a particle around a closed path C is $\oint_c \vec{F}\cdot\overrightarrow{dr} = 0$

Example 16: If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, then evaluate the line integral $\oint_C \vec{F}.d\vec{r}$ along the curve C in the x-y plane given by $y = x^3$ from the point (1,1) to (2,8).

Soln: $\vec{r} = x\,\hat{i} + x^3\hat{j} \Rightarrow d\vec{r} = dx\,\hat{i} + 3x^2 dx\,\hat{j}$

$$So, \int_C \vec{F}.d\vec{r} = \int_1^2 \left[(5x^4 - 6x^2)\hat{i} + (2x^3 - 4x)\hat{j}\right].\left[dx\,\hat{i} + 3x^2 dx\,\hat{j}\right]$$

$$= \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx = 35$$

Example 17: If a force field $\vec{A} = y\hat{i} - x\hat{j}$ is applied to a particle, then the work done by the force field in traversing the particle around a circle C in x-y plane, with center at the origin and radius 1 units is equal to (C is traversed in the counter-clockwise direction)

(a) 2π (b) -2π (c) 8π (d) -8π

Soln: $\oint_C \vec{A}.d\vec{r} = \oint_C [ydx - xdy] \Rightarrow$ Putting $x = \cos\theta$ and $y = \sin\theta$ we get,

$$= \int_0^{2\pi} [\sin\theta(-\sin\theta d\theta) - \cos\theta(\cos\theta d\theta)] = -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = -2\pi$$

Correct option is (b)

Example 18: Evaluate $\int_C \vec{F}.\overrightarrow{dr}$, where $F = x^2\hat{i} + y^3\hat{j}$ and curve C is the arc of parabola $y = x^2$ in the x-y plane from (0, 0) to (1, 1).

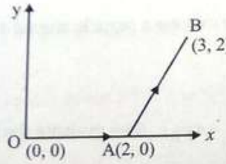
Soln: Along the curve C, $y = x^2 \Rightarrow dy = 2xdx$

Therefore, $\int_C \vec{F}.d\vec{r} = \int_{x=0}^{1}\left[x^2dx + x^6(2x)dx\right]$

$$= \int_0^1 \left(x^2 + 2x^7\right)dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}$$

Example 19: If $\vec{F} = (2x + y)\hat{i} + (3y + x)\hat{j}$, evaluate $\int\limits_C \vec{F}.\overrightarrow{dr}$ where C is the curve in the xy-plane consisting of the straight lines from (0, 0) to (2, 0) and then to (3, 2).

Soln: The path of integration C has been shown in the figure below:



We have, $\int_C \vec{F}.d\vec{r} = \int_C \left[(2x + y)\hat{i} + (3y - x)\hat{j}\right].\left(dx\,\hat{i} + dy\,\hat{j}\right)$

$$= \int_C \left[(2x + y)dx + (3y - x)dy\right]$$

Path 1: Along the straight line OA, y = 0, dy = 0 and x varies from 0 to 2.

Path 2: Equation of the straight line AB is $y - 0 = \frac{2-0}{3-2}(x - 2)$ *i.e.* $y = 2x - 4$

Therefore, along AB, y = 2x–4, dy = 2dx and x varies from 2 to 3.

$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_0^2\left[(2x + 0)dx + 0\right] + \int_2^3\left[(2x + 2x - 4)dx + (6x - 12 - x)2dx\right]$$

$$= \left[x^2\right]_0^2 + \int_0^2 (14x - 28)dx = 4 + 14\int_2^3 (x - 2)dx = 11$$

Example 20: Find the total work done in moving a particle in a force field given by $F = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Soln: The total work done

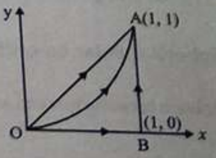
$$= \int_C \vec{F}.d\vec{r} = \int_C \left(3xy\hat{i} - 5z\hat{j} + 10x\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \int_1^2 (3xydx - 5zdy + 10xdz)$$

$$= \int_1^2\left[3(t^2 + 1)(2t^2)(2t) - (5t^3)(4t) + 10(t^2 + 1)(3t^2)\right]dt = \int_1^2\left(12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2\right)dt = 303$$

Example 21: If $F = y\hat{i} - x\hat{j}$, evaluate $\int_C \vec{F}.\overrightarrow{dr}$ from (0, 0) to (1, 1) along the following path C.

(a) the parabola $y = x^2$
(b) the straight lines from (0, 0) to (1, 0) and then to (1, 1)
(c) the straight line joining (0, 0) and (1, 1)

Soln: The three paths of integration have been shown in the figure



We have $\int_C \vec{F}.d\vec{r}$

$$= \int_C (yi + xj).(dxi + dyj) = \int_C (ydx - xdy)$$

(a) C is the arc of parabola $y = x^2$ from (0, 0) to (1, 1)
Here $dy = 2xdx$ and x varies from 0 to 1.

$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_0^1\left[x^2dx - x(2x)dx\right] = \int_0^1 -x^2dx = -\frac{1}{3}$$

(b) C is the curve consisting of straight lines OB and BA.
Along OB, y = 0, dy = 0 and x varies from 0 to 1
Along BA, x = 1, dx = 0 and y varies from 0 to 1

$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_0^1 0\,dx + \int_0^1 -1\,dy = -1$$

(c) C is the straight lines OA. The equaiton of OA is

$$y - 0 = \frac{1-0}{1-0}(x - 0) \; i.e. \; y = x$$

Therefore, dy = dx and x varies from 0 to 1.

$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_0^1 (xdx - xdx) = 0$$

Example 21: Evaluate $\int_C \left\{\left(2xy^3 - y^2\cos x\right)dx + \left(1 - 2y\sin x + 3x^2y^2\right)dy\right\}$

Where C is the arc of the parabola $2x = \pi y^2$ from (0, 0) to $\left(\frac{1}{2}\pi, 1\right)$

Soln: The given integral in the question is the line integral of the following vector field:

$$\vec{F} = \left(2xy^3 - y^2\cos x\right)\hat{i} + \left(1 - 2y\sin x + 3x^2y^2\right)\hat{j}$$

This vector field is conservative nature as $\vec{\nabla} \times \vec{F} = 0$ and it can be written as $\vec{F} = \vec{\nabla}\phi$

$$\frac{\partial\phi}{\partial x} = 2xy^3 - y^2\cos x \Rightarrow \phi = x^2y^3 - y^2\sin x + f_1(y)$$

$$\frac{\partial\phi}{\partial y} = \left(1 - 2y\sin x + 3x^2y^2\right) \Rightarrow \phi = y - y^2\sin x + x^2y^3 + f_2(x)$$

Therefore, $\phi = y - y^2\sin x - x^2y^3 + c$

Since, work done by a conservative field i.e value of the given integral is independent of path and will be equal to

$$I = \phi\left(\frac{\pi}{2}, 1\right) - \phi(0,0) = \left[y - y^2\sin x + x^2y^3\right]_{(0,0)}^{(\pi/2,1)} = \frac{\pi^2}{4}$$

1.5 Orthogonal Currvilinear Co-ordinates

Spherical Polar co-ordinates (r, θ, ϕ):

Relation between cartesian and spherical polar co-ordiantes:

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

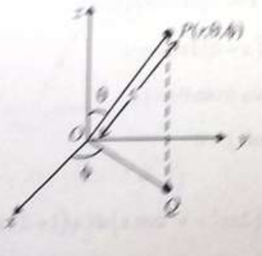
In spherical polar co-ordinates,

r = radial distance of the point P from the origin

θ = angle which OP makes with z-axis

ϕ = angle which OQ (projection of OP in x-y plane) makes with positive x-axis

Corresponding limits: $0 \le r < \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$



Elementary line elements:

Line element between r to $r+dr$: $d\vec{l}_r = dr\,\hat{r}$

Line element between θ to $\theta+d\theta$: $d\vec{l}_\theta = r\,d\theta\,\hat{\theta}$

Line element between ϕ to $\phi+d\phi$: $d\vec{l}_\phi = r\sin\theta\,d\phi\,\hat{\phi}$

Elementary surface elements:

Constant r surface : $d\vec{s}_r = d\vec{l}_\theta \times d\vec{l}_\phi = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$ (surface of a sphere)

Constant θ surface : $d\vec{s}_\theta = d\vec{l}_\phi \times d\vec{l}_r = r\sin\theta\,dr\,d\phi\,\hat{\theta}$

Constant ϕ surface : $d\vec{s}_\phi = d\vec{l}_r \times d\vec{l}_\theta = r\,dr\,d\theta\,\hat{\phi}$

Elementary volume element : $dV = d\vec{l}_r.\left(d\vec{l}_\theta \times d\vec{l}_\phi\right) = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$

Divergence: $\vec{\nabla}.\vec{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, a_\theta) + \frac{1}{r^2\sin\theta}\frac{\partial a_\phi}{\partial\phi}$

Curl: $\vec{\nabla}\times\vec{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ a_r & r a_\theta & r\sin\theta a_\phi \end{vmatrix}$

Laplacian: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}$

Unit vectors: $\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\hat{k}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\hat{k}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Suppose the vector is given in cartesian coordinates i.e. $\vec{r} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and we have to convert it in sphereical polar coordinates i.e $\vec{r} = a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\phi}$, the components are related by the following equation.

$$\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Limits for various types of integrations:

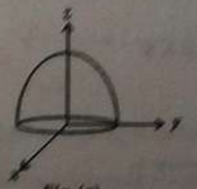


Fig. (a)
Upper hemisphere

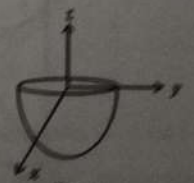


Fig. (b)
Lower hemisphere

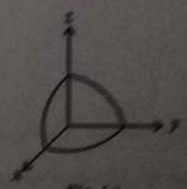


Fig. (c)
First Octant of sphere

Case-1: Full sphere : $x^2 + y^2 + z^2 = a^2$

Limit of volume integration : $0 \le r \le a;\ 0 \le \theta \le \pi;\ 0 \le \phi \le 2\pi$

Limit of surface integration : $r = \text{const} = a;\ 0 \le \theta \le \pi;\ 0 \le \phi < 2\pi$

Case-II: Upper Hemisphere : $x^2+y^2+z^2=a^2$

Limit of volume integration : $0 \le r \le a;\ 0<\theta \le \frac{\pi}{2};\ 0 \le \phi \le 2\pi$

Limit of surface integration : $r=\text{const}=a;\ 0 \le \theta \le \frac{\pi}{2};\ 0 \le \phi < 2\pi$

Case-III: Lower hemisphere : $x^2+y^2+z^2=a^2$

Limit of volume integration : $0 \le r \le a;\ \frac{\pi}{2}<\theta \le \pi;\ 0 \le \phi \le 2\pi$

Limit of surface integration : $r=\text{const}=a;\ \frac{\pi}{2} \le \theta \le \pi;\ 0 \le \phi < 2\pi$

Case-IV: First octant of sphere: $x^2+y^2+z^2=a^2$

Limit of volume integration : $0 \le r \le a;\ 0<\theta \le \frac{\pi}{2};\ 0 \le \phi \le \frac{\pi}{2}$

Limit of surface integration : $r=\text{const}=a;\ 0 \le \theta \le \frac{\pi}{2};\ 0 \le \phi < \frac{\pi}{2}$

Cylindrical co-ordinates (ρ, ϕ, z):

Relation between cartesian and cyllindrical co-ordiantes:

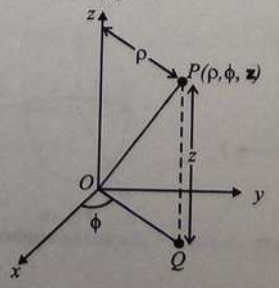
$$x=\rho\cos\phi,\ y=\rho\sin\phi,\ z=z$$

In cyllindrical co-ordinates,

ρ = perpendicular distance point 'P' from the z axis

ϕ = angle which OQ (projection of OP in x-y plane) makes with positive x-axis

z = height of the point 'P' above the x-y plane



Elementary line elements:

Line element between ρ to $\rho+d\rho$: $d\vec{l}_\rho = d\rho\,\hat{\rho}$

Line element between ϕ to $\phi+d\phi$: $d\vec{l}_\phi = \rho d\phi\,\hat{\phi}$

Line element between z to $z+dz$: $d\vec{l}_z = dz\,\hat{z}$

Elementary surface elements:

Constant ρ surface : $d\vec{s}_\rho = d\vec{l}_\phi \times d\vec{l}_z = \rho d\phi dz\,\hat{\rho}$ (curved surface of a cyllinder)

Constant ϕ surface : $d\vec{s}_\phi = d\vec{l}_z \times d\vec{l}_r = d\rho dz\,\hat{\phi}$

Constant z surafce : $d\vec{s}_z = d\vec{l}_\rho \times d\vec{l}_\phi = \rho d\rho d\phi\,\hat{z}$ (plane surface of a cyllinder parallel to x - y plane)

Elementary volume element : $dV = d\vec{l}_\rho \cdot \left(d\vec{l}_\phi \times d\vec{l}_z\right) = \rho d\rho d\phi dz$

Gradient: $\vec{\nabla} = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}.\vec{a} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{a} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix}$

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$

Unit Vectors: $\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\hat{z} = \hat{k}$

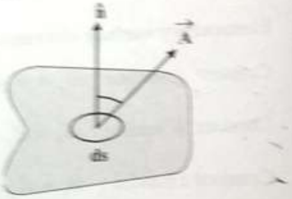
Suppose the vector is given in cartesian coordinates i.e. $\vec{r} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and we have to convert it in sphereical polar coordinates i.e $\vec{r} = a_\rho\hat{\rho} + a_\phi\hat{\phi} + a_z\hat{z}$, the components are related by the following equation.

$$\begin{bmatrix} a_\rho \\ a_\phi \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

1.6 Surface Integration of Vectors

The integration of vector on an open or closed surface is known as surface integration of vectors.

Let $\bar{A}$ be a vector function and S be the given surface. Surface integral of a vector function $\bar{A}$ over the surface S is defined as the integral of the components of $\bar{A}$ along the normal to the surface.



Components of $\bar{A}$ along the normal

$=\bar{A}.\hat{n}$, where $\hat{n}$ is the unit normal vector to an element ds

Surface integral of $\bar{A}$ over S

$$=\Sigma\bar{A}.\hat{n}=\iint_S\left(\bar{A}.\hat{n}\right)ds$$

Physical interpetation:

$\iint_S \bar{A}\cdot\bar{ds}$ represents the flux of the vector function $\bar{A}$ through the surface S.

If the surface of integration 'S' is not parallel to any one of the co-ordinate planes x-y plane, y-z plane or x-z plane, then we have to draw projection of the surface element $\bar{ds}$ in any one of the co-ordinate planes and the above surface integral will be replaced as follows:

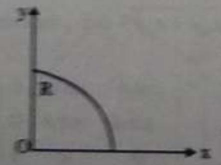
If the projection is drawn in x-y plane then $\iint_S \bar{A}.\bar{ds}=\iint_{x-y}\bar{A}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{k}|}$

If the projection is drawn in y-z plane then $\iint_S \bar{A}.\bar{ds}=\iint_{y-z}\bar{A}.\hat{n}\frac{dydz}{|\hat{n}.\hat{i}|}$

If the projection is drawn in x-z plane then $\iint_S \bar{A}.\bar{ds}=\iint_{x-z}\bar{A}.\hat{n}\frac{dxdz}{|\hat{n}.\hat{j}|}$

Example 22: Evaluate $\iint_S \bar{F}.\hat{n}\bar{ds}$, where $\bar{F}=yz\,\hat{i}+zx\,\hat{j}+xy\,\hat{k}$ and S is that part of the surface of the sphere $x^2+y^2+z^2=1$ while lies in the first octant.

Soln: A vector normal to the surface S is given by $\hat{n}=\frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{(4x^2+4y^2+4z^2)}}=x\hat{i}+y\hat{j}+z\hat{k}$



Since, $x^2+y^2+z^2=1$ on the surface S.

We have $\iint_S \bar{F}.\hat{n}\bar{ds}=\iint_R \bar{F}.\hat{n}\frac{dx\,dy}{|\hat{n}.\hat{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the circle $x^2+y^2=1, z=0$.

We have, $\bar{F}.\hat{n}=\left(yz\hat{i}+zx\hat{j}+xy\hat{k}\right).\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=3xyz$ and $\hat{n}.\hat{k}=\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\hat{k}=z$

Hence, $\iint_S \bar{F}.\hat{n}ds=\iint_R\frac{3xyz}{z}dxdy=3\iint_R xy\,dx\,dy\;=3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(r\cos\theta)(r\sin\theta)r\,d\theta\,dr$

$$=3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1\cos\theta\sin\theta\,d\theta=\frac{3}{4}\left(\frac{1}{2}\right)=\frac{3}{8}$$

Example 23: Evaluate $\iint_S \bar{F}.\hat{n}\bar{ds}$, where $\bar{F}=z\hat{i}+x\hat{j}-3y^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between z = 0 and z = 5

Soln: A unit vector normal to the surface $S\,(x^2+y^2=16)$ is $\hat{n}=\frac{2x\hat{i}+2y\hat{j}}{\sqrt{(4x^2+4y^2)}}=\frac{x\hat{i}+y\hat{j}}{4}$

We have $\iint_S \bar{F}.\hat{n}dS=\iint_R \bar{F}.\hat{n}\frac{dx\,dz}{|\hat{n}.\hat{j}|}$, where R is the projection of S on the x-z plane. It should be noted that in this case we cannot take the projection of S on the x-y plane as the surface S is perpendicular to the x-y plane.

Now, $\bar{F}.\hat{n}=\left(z\hat{i}+x\hat{j}-3y^2z\hat{k}\right)\left(\frac{x\hat{i}+y\hat{j}}{4}\right)=\frac{1}{4}(xz+xy)$ and $\hat{n}.\hat{j}=\left(\frac{x\hat{i}+y\hat{j}}{4}\right).\hat{j}=\frac{y}{4}$

Therefore, the required surface integral is

$$I=\iint_R\frac{xz+xy}{4}\frac{dx\,dz}{y/4}=\int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dx\,dz=90$$

Example 24: Evaluate $\iint_S F.\hat{n}\bar{ds}$, where $\bar{F}=\left(x+y^2\right)\hat{i}-2x\hat{j}+2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

Soln: A unit vector normal to the surface S is $\hat{n}=\frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{(4+1+4)}}=\left(\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$

We have $\iint_S \bar{F}.\hat{n}\,dS=\iint_R F.n\frac{dx\,dz}{|\hat{n}.\hat{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the straight line 2x+y=6, z=0

We have, $\vec{F}.\hat{n}=\left[\left(x+y^2\right)\hat{i}-2x\hat{j}+2yz\hat{k}\right]\left(\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)=\frac{2}{3}y^2+\frac{4}{3}yz$

Also, $n.\hat{k}=\left(\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right).\hat{k}=\frac{2}{3}$

Hence, $\iint_R\left(y^2+2yz\right)dx\,dy=\iint_R\left(y^2+2yz\right)dx\,dy=\iint_R\left[y^2+2y\left(\frac{6-2x-y}{2}\right)\right]dx\,dy$

$$=\iint_R\left(y^2+6y-2xy-y^2\right)dx\,dy=2\iint_R y(3-x)\,dx\,dy=2\int_{y=0}^{6}\int_{x=0}^{(6-y)/2}y(3-x)\,dx\,dy$$

[Note that R is bounded by x-axis, y-axis and the straight line $2x+y=6,\ z=0$. To evaluate the double integral over R, keep y fixed and integrate with respect to x from $x=0$ to $x=\frac{6-y}{2}$, then integrate with respect to y from y = 0 to y = 6.]

$$=2\int_0^6 y\left[3x-\frac{x^2}{2}\right]_{x=0}^{(6-y)/2}dy=2\int_0^6 y\left[\frac{9}{2}-\frac{y^2}{8}\right]_0^6 dy=\int_0^6\left[9y-\frac{y^3}{4}\right]dy=81$$

Example 25: Evaluate $\iint_S F.\hat{n}\overline{ds}$, where $\vec{F}=y\hat{i}+2x\hat{j}-z\hat{k}$ and S is the surface of the surface of the plane $2x+y=6$ in the first octant cut off by the plane z = 4

Soln: A unit vector normal to the surface S is $\hat{n}=\frac{2\hat{i}+\hat{j}}{\sqrt{(4+1)}}=\frac{1}{\sqrt{5}}\left(2\hat{i}+\hat{j}\right)$

We have $\iint_S F.ndS=\iint_R\bar{F}.\hat{n}\frac{dx\,dz}{\left|\hat{n}.\hat{j}\right|}$, where R is the projection of S on the xz-plane. It should be noted that in this case we cannot take the projection on the xy-plane because the surface S is perpendicular to xy-plane.

Now, $\bar{F}.\hat{n}=\left(y\hat{i}+2x\hat{j}-z\hat{k}\right)\left(\frac{2}{\sqrt{5}}\hat{i}+\frac{1}{\sqrt{5}}\hat{j}\right)=\frac{2}{\sqrt{5}}y+\frac{2}{\sqrt{5}}x$ and $\hat{n}.\hat{j}=\frac{1}{\sqrt{5}}\left(2\hat{i}+\hat{j}\right).\hat{j}=\frac{1}{\sqrt{5}}$

Therefore, the required surface integral is

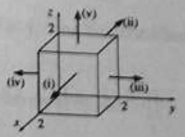
$$=\iint_R\left(\frac{2y}{\sqrt{5}}+\frac{2}{\sqrt{5}}x\right)\sqrt{5}dx\,dz=\iint_R 2(y+x)\,dx\,dz=2\iint_R(6-2x+x)\,dxdz$$

$$=2\int_{x=0}^{3}(6-x)[z]_0^4\,dx=8\left[6x-\frac{x^2}{2}\right]_0^3=108$$

Example 26: Calculate the surface integral of $\vec{v}=2xz\,\hat{x}+(x+2)\,\hat{y}+y\left(z^2-3\right)\hat{z}$ over five sides (excluding the bottom) of the cubical box shown in figure.

Soln: (i) For the surface $x=2,\ d\bar{s}=dy\,dz\,\hat{x},\quad \vec{v}.d\bar{s}=2xzdy\,dz=4z\,dy\,dz,$ So, $\int\vec{v}.d\vec{s}=0$

(ii) For the surface $x=0,\ d\bar{s}=-dy\,dz\,\hat{x},\quad \vec{v}.d\bar{s}=-2xzdy\,dz=0,$ so $\int\vec{v}.d\vec{s}=0$



(iii) For the surface $y=2,\ d\bar{s}=dx\,dz\,\hat{y},\quad \vec{v}.d\bar{s}=(x+2)\,dx\,dz,$ so $\int\vec{v}.d\bar{s}=\int_0^2(x+2)\,dx\int_0^2 dz=12$

(iv) For the surface $y=0,\ d\bar{s}=-dx\,dz\,\hat{y},\quad \vec{v}.d\bar{s}=-(x+2)\,dx\,dz,$ so $\int\vec{v}.d\bar{s}=-\int_0^2(x+2)\,dx\int_0^2 dz=-12$

(v) For the surface $z=2,\ d\bar{s}=dxdy\,\hat{z},\quad \vec{v}.d\bar{s}=ydxdy,$ so $\int\vec{v}.d\bar{s}=\int_0^2 dx\int_0^2 ydy=4$

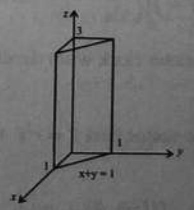
Total flux $=\int_{surface}\vec{v}.d\bar{s}=16+0+12-12+4=20$

1.7 Volume Integration of Vectors

Volume integral of a vector field $\vec{A}$ over volume V can be written as,

$\iiint_V\vec{A}dV$, where, dV is the infinitesimal volume element

Example 27: Calculate the volume integral of $T=xyz^2$ over the prism in figure.



Soln: $\int Tdv=\int_0^3 z^2\left\{\int_0^1 y\left[\int_0^{1-y}xdx\right]dy\right\}dz=\frac{1}{2}\int_0^3 z^2dz\int_0^1(1-y)^2\,y\,dy=\frac{1}{2}(9)\left(\frac{1}{12}\right)=\frac{3}{8}$

Example 28: The volume of the portion of the cyllinder $x^2+y^2=4$ in the first octant between the planes z = 0 and $3x$ - z = 0 is

(a) 2 (b) 4 (c) 8 (d) 16

Soln: Volume of the portion of the cyllinder in the first octant $=V=\int_0^2\int_0^{\sqrt{4-x^2}}\int_{z=0}^{3x}dxdydz=8$

1.8 Vector Related Theorems

Divergence Theorem:

This theorem is applicable only for closed surfaces and this theorem converts surface integral into volume integral and vice versa.

If V is the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivatives, then.

$$\iiint_V (\vec{\nabla}\cdot\vec{A})dv = \oiint_S \vec{A}\cdot\overline{ds} = \oiint_S \vec{A}\cdot\hat{n}ds$$

where $\hat{n}$ is the outward normal to 'S'

Stoke's Theorem:

This theorem is applicable only for open surfaces and this theorem converts surface integral into line integral and vice versa.

If S is an open, two-sided surface bounded by a closed, non-intersecting curve C, and $\vec{A}$ is a vector function of position with continuous derivatives; then

$$\oint_C \vec{A}\cdot\overline{dr} = \iint_S (\vec{\nabla}\times\vec{A}).\hat{n}ds = \iint_S (\vec{\nabla}\times\vec{A})\cdot\overline{ds}$$

where C is traversed in the positive (counterclockwise) direction.

Green's Theorem:

If R is a closed region of the x-y plane, bounded by a simple closed curve C and M, N are continuous functions of x and y having continuous derivatives in R, then.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

where C is traversed in the positive (counter clock wise) direction.

Example 29: Calculate the flux of the vector field $\vec{V} = r^2\vec{r}$ through the surface S of a sphere of radius 'a', centered at the origin.

Soln: Flux of the vector field $\vec{V} = \oiint_S \vec{V}.d\vec{s} = \iiint_V (\vec{\nabla}.\vec{V})dv$ (Using divergence theorem)

$$= \iiint_V (\vec{\nabla}.r^2\vec{r})dv = \iiint_V 5r^2 dv = 5\iiint_V r^2(r^2 \sin\theta drd\theta d\phi) = 5\int_0^a r^4 dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi = 4\pi a^5$$

Example 30: Calculate $\oint_C \vec{A}.d\vec{r}$ for the vector field $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ along the boundary of upper half of the sphere $x^2+y^2+z^2=1$.

Soln: The boundary is the circle of unit radius in the x-y plane i.e. $x^2+y^2=1$. Therefore, substituting $x=\cos\theta$, $y=\sin\theta$ and $z=0$ we get,

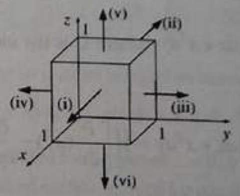
$$\oint_C \vec{A}.d\vec{r} = \oint_C\left[(2x-y)dx - yz^2dy - y^2zdz\right] = \oint_C (2x-y)dx = \int_0^{2\pi}(2\cos\theta-\sin\theta)(-\sin\theta d\theta) = \pi$$

Example 31: Check the divergence theorem using the function

$$\vec{A} = y^2\hat{x} + (2xy+z^2)\hat{y} + (2yz)\hat{z}$$

over the unit cube situated at the origin shown in figure.

Soln:



$$\iiint_V (\vec{\nabla}.\vec{A})dv = \iiint_V 2(x+y)dv = 2\int_0^1\int_0^1\int_0^1 (x+y)dx\,dy\,dz$$

$$= 2\left[\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz + \int_0^1\int_0^1\int_0^1 y\,dx\,dy\,dz\right] = 2\left[\frac{1}{2}\times1\times1 + \frac{1}{2}\times1\times1\right] = 2$$

To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $\int\vec{A}.d\vec{s} = \int_0^1\int_0^1 y^2dy\,dz = \frac{1}{3}$

(ii) $\int\vec{A}.d\vec{s} = -\int_0^1\int_0^1 y^2dy\,dz = -\frac{1}{3}$

(iii) $\int\vec{A}.d\vec{s} = -\int_0^1\int_0^1 (2x+z^2)dx\,dz = \frac{4}{3}$

(iv) $\int\vec{A}.d\vec{s} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\frac{1}{3}$

(v) $\int\vec{A}.d\vec{s} = -\int_0^1\int_0^1 2y\,dx\,dy = 1$

(vi) $\int\vec{A}.d\vec{s} = -\int_0^1\int_0^1 0\,dx\,dy = 0$

So the total flux is:

$\oint_S \vec{A}.d\vec{s} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$. Hence, divergence theorem is verified.

Example 32: Suppose $\vec{A} = (2xz+3y^2)\hat{j} + (4yz^2)\hat{k}$. Check Stokes' theorem for the square surface shown in figure.

Soln: Here, $\nabla\times\vec{A} = (4z^2-2x)\hat{i} + 2z\,k$ and $d\vec{s} = dy\,dz\,\hat{i}$

Since $x=0$ for this surface, $\int(\nabla\times\vec{A})\overline{ds} = \int_0^1\int_0^1 4z^2dy\,dz = \frac{4}{3}$

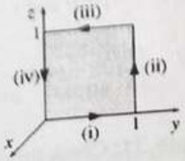
Now, for line integral, We must break this up into four segments:

(i) $x=0, z=0, \vec{A}.\overrightarrow{dl}=3y^2dy \Rightarrow \int \vec{A}.\overrightarrow{dl}=\int_0^1 3y^2dy=1$

(ii) $x=0, y=1, \vec{A}.\overrightarrow{dl}=4z^2dz \Rightarrow \int \vec{A}.\overrightarrow{dl}=\int_0^1 4z^2dz=\frac{4}{3}$

(iii) $x=0, z=1, \vec{A}.\overrightarrow{dl}=3y^2dy \Rightarrow \int \vec{A}.\overrightarrow{dl}=\int_1^0 3y^2dy=-1$

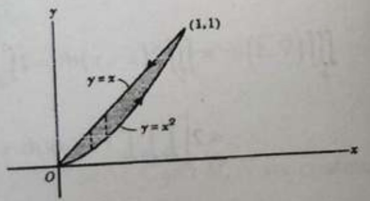
(iv) $x=0, y=0, \vec{A}.\overrightarrow{dl}=0 \Rightarrow \int \vec{A}.\overrightarrow{dl}=\int_1^0 0dz=0$



So, $\oint \vec{A}.\overrightarrow{dl}=1+\frac{4}{3}-1+0=\frac{4}{3}$

Example 34: Evaluate $\oint_C (xy+y^2)dx+x^2dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Soln: Using green's theorem, $\oint_C (xy+y^2)dx+x^2dy=\iint_R \left[\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(xy+y^2)\right]dxdy$



$$=\iint_R (x-2y)dxdy=\int_{x=0}^1\int_{y=x^2}^{x}(x-2y)dydx$$

$$=\int_0^1\left[\int_{x^2}^{x}(x-2y)dy\right]dx=\int_0^1 (xy-y^2)\Big|_{x^2}^{x}dx$$

$$=\int_0^1 (x^4-x^3)dx=-\frac{1}{20}$$

Example 34: Calculate the flux of the vector field $\vec{F}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ through the surface bounded by the region $x^2+y^2=4, z=0, z=3$.

Soln: Flux of the vector field $=\oiint_S \vec{F}.\vec{ds}=\iiint_V (\vec{\nabla}.\vec{F})dv$ (according to divergence theorem)

$$=\iiint_V (4-4y+2z)dxdydz=\int_{z=0}^{3}\int_{\phi=}^{2\pi}\int_{\rho=0}^{2}(4-4\rho\sin\phi+2z)\rho d\rho d\phi dz=84\pi$$

Example 35: Evaluate $\oint_C \left[(2x^2-y^2)dx+(x^2+y^2)dy\right]$, where 'C' is the boundary of the surface enclosed by x-axis and the semicircle $y=(1-x^2)^{1/2}$.

Soln: $\oint_C \left[(2x^2-y^2)dx+(x^2+y^2)dy\right]=\oiint_{x-y}[2x+2y]dydx$ (according to green's theorem)

$$=\int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}}[2x+2y]dydx=2\int_{x=-1}^{1}\left[xy+\frac{y^2}{2}\right]_{y=0}^{\sqrt{1-x^2}}dx=2\left[\int_{x=-1}^{1}x\sqrt{1-x^2}dx+\int_{-1}^{1}\frac{(1-x^2)}{2}dx\right]=\frac{4}{3}$$

Example 36: Evalaute by Green's theorem

$$\oint_C (x^2-\cosh y)dx+(y+\sin x)dy$$

where C is the rectangle with vertices $(0,0),(\pi,0),(\pi,1),(0,1)$

Soln: Using Green's Theorem, the given line integral will be

$$\iint_R (\cos x+\sinh y)dxdy=\int_{x=0}^{\pi}\int_{y=0}^{1}(\cos x+\sinh y)dxdy$$

$$=\int_{x=0}^{\pi}[y\cos x+\cosh y]_{y=0}^{1}dx=\int_{x=0}^{\pi}[\cos x+\cosh-1]dx=[\sin x+x\cosh 1-x]_0^{\pi}=\pi(\cosh 1-1)$$

Example 37: By transforming to a triple integral evaluate

$$I=\iint_S (x^3dydz+x^2ydzdx+x^2z\,dx\,dy)$$

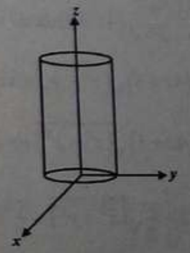
where S is the closed surface bounded by the planes $z=0, z=b$ and the cylinder $x^2+y^2=a^2$.

Soln. By divergence theorem, the required surface integral I is equal to the volume integral.

$$\iiint_V \left[\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(x^2y)+\frac{\partial}{\partial z}(x^2z)\right]dV$$

$$=\iiint 5x^2dxdydz=5\iiint \rho^2\cos^2\phi(\rho d\rho d\phi dz)$$

$$=5\int_{\rho=0}^{a}\rho^3d\rho\int_{\phi=0}^{2\pi}\cos^2\phi d\phi\int_{z=0}^{b}dz=5.\frac{a^4}{4}.\pi b=\frac{5}{4}\pi ba^4$$



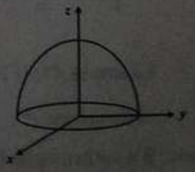
Example 38: Evaluate, $\iint_S (y^2z^2\hat{i}+z^2x^2\hat{j}+z^2y^2\hat{k}).\hat{n}\,ds$

Where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane are bounded by this plane.

Soln: By divergence theorem, we have

$$\iint_S (y^2z^2\hat{i}+z^2x^2\hat{j}+z^2y^2\hat{k}).\hat{n}ds=\iiint_V div(y^2z^2\hat{i}+z^2x^2\hat{j}+z^2y^2\hat{k})dV$$

$$=\iiint_V 2zy^2\,dV=2\iiint_V zy^2dV$$



Using spherical polar coordinates (r,θ,ϕ), the integral will be

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)(r^2\sin^2\theta\sin^2\phi)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}r^5\sin^3\theta\cos\theta\sin^2\phi\,dr\,d\theta\,d\phi=\frac{\pi}{2}$$

Example 39: By converting the surface integral into a volume integral evaluate

$$\iint_S \left(x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dx\, dy\right)$$

Where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$

Soln: Using divergence theorem, $\iint_S \left(x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dx\, dy\right) = \iiint_V 3\left(x^2 + y^2 + z^2\right) dx dy dz$

$$= 3\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 r^2 \sin\theta dr\, d\theta\, d\phi = 3\times 2\pi\times 2\times\frac{1}{5} = \frac{12\pi}{5}$$

Example 40: Evaluate $\iint_S \vec{F}.\hat{n}dS$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, given $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

Soln: By divergence theorem, we have

$$\iint_S \vec{F}.\hat{n}dS = \iiint_V div\vec{F}\, dV = \iiint_V \left(4z + xz^2 + 3\right) dx\, dy\, dz$$

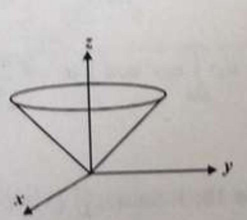
$$= \int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{(z^2-y^2)}} \left(4z + xz^2 + 3\right) dx\, dy\, dz$$

$$= \int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{(z^2-y^2)}} (4z+3)\, dx\, dy\, dz \qquad \left[\text{since, } \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{(z^2-y^2)}} x dx = 0\right]$$

$$= 2\int_{z=0}^{4}\int_{y=-z}^{z} (4z+3)\sqrt{\left(z^2-y^2\right)} dydz$$

$$= 2\int_{z=0}^{4}\int_{y=-z}^{z} (4z+3)\sqrt{\left(z^2-y^2\right)}\, dy\, dz$$

$$= 2\int_{z=0}^{4} (4z+3)\left[\frac{y}{2}\sqrt{\left(z^2-y^2\right)} + \frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_{-z}^{z} dz$$



$$= 4\int_0^4 (4z+3)\left[\frac{z^2}{2}\sin^{-1}1\right] dz = \pi\int_0^4 \left(4z^3 + 3z^2\right) dz$$

$$= 4\left[z^4 + z^3\right]_0^4 = \pi\left(256 + 64\right) = 320\pi$$

Example 41: If $F = \left(x^2 + y - 4\right)\hat{i} + 3xy\hat{j} + \left(2xz + z^2\right)\hat{k}$, evaluate $\iint_S \left(\nabla\times\vec{F}\right).\hat{n}dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 16$ above the xy-plane.

Soln: S is the hemispherical surafce $x^2 + y^2 + z^2 = 16$ bounded by the circle C given by $x^2 + y^2 = 16,\ z = 0$.

Using Stoke's theorem,

$$\iint_S \left(\vec{\nabla}\times\vec{F}\right).\hat{n}dS = \oint_C \vec{F}.d\vec{r} = \oint_C \left(x^2 + y - 4\right)dx + 3xydy$$

Putting $x = 4\cos\theta$ and $y = 4\sin\theta$, we get

$$= \int_{\theta=0}^{2\pi} \left(16\cos^2\theta + 4\sin\theta - 4\right)\left(-4\sin\theta d\theta\right) + 3\left(4\cos\theta\right)\left(4\sin\theta\right)\left(4\cos d\theta\right) = -16\pi$$

Example 42: Evaluate by Stoke's theorem

$$\oint_C \left(e^x dx + 2ydy - dz\right)$$

where C is the curve $x^2 + y^2 = 4,\ z = 2$

Soln. $\oint_C \left(e^x dx + 2y\, dy - dz\right) = \oint_C \left(e^x\hat{i} + 2y\hat{j} - \hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \oint_C \vec{F}.d\vec{r}$, where $F = e^x\hat{i} + 2y\hat{j} - \hat{k}$

$$\text{Also, } curl\ F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0$$

Hence, $\oint_C \vec{F}.d\vec{r} = \iint_S \left(curl\ \vec{F}\right).\hat{n}\ dS = 0$, since curl F=0

Example 43: Use Stoke's theorem to evaluate $\iint_S \left(\nabla\times\vec{F}\right).\hat{n}dS$, where $F = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane

Soln: The boundary C of the surface S is the circle $x^2 + y^2 = a^2,\ z = 0$.

Using stoke's theorem, we have

$$= \iint_S \left(\nabla\times\vec{F}\right).\hat{n}dS = \int_C F.d\vec{r} = \int_S \left[y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}\right].\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \int_C \left[ydx + (x - 2xz)dy - xy\, dz\right] = \int_C \left(ydx + xdy\right) \qquad [\because \text{on } C,\ z = 0 \text{ and } dz = 0]$$

Putting $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$, we get

$$= \int_0^{2\pi} \left[a\sin t\left(-a\sin t\right) + a\cos t\left(a\cos t\right)\right] dt = 0$$

1.9 Linear Dependency of Vectors

Three vectors $\vec{u}, \vec{v}, \vec{w}$ will be linearly dependent if $\alpha_1\vec{u} + \alpha_2\vec{v} + \alpha_3\vec{w} = 0$, where $\alpha_1, \alpha_2, \alpha_3$ are not all zero. These three independent vectors do not form the basis of $\mathbb{R}^3$ space.

If $\alpha_1\vec{u} + \alpha_2\vec{v} + \alpha_3\vec{w} = 0$ such that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $\vec{u}, \vec{v}, \vec{w}$ are said to be linearly independent and they do form the basis of $\mathbb{R}^3$ space i.e. any vector in $\mathbb{R}^3$ space can be written as the linear combination of vectors $\vec{u}, \vec{v}$, w respectively.

How to check:

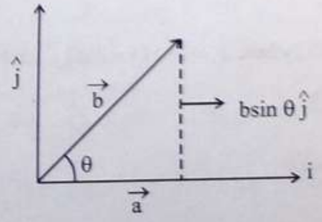
$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \begin{cases} = 0 \text{ for linearly dependent vectors} \\ \neq 0 \text{ for linearly independent vectors} \end{cases}$$

CSIR-NET/JRF Previous Years Questions with Solutions

1. Let $\vec{a}$ and $\vec{b}$ be two distinct three-dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by **[CSIR June - 2011]**

(a) $\dfrac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$ (b) $\dfrac{\vec{b}\times(\vec{a}\times\vec{b})}{b^2}$ (c) $\dfrac{(\vec{a}\vec{b})\vec{b}}{b^2}$ (d) $\dfrac{(\vec{b}.\vec{a})\vec{a}}{a^2}$

Soln.



Assume that $\vec{a}$ and $\vec{b}$ lie in the xy plane, Component of $\vec{b} \perp \vec{a} = (b\sin\theta)\,\hat{j}$

$$\vec{a}\times\vec{b} = ab\sin\theta(\hat{i}\times\hat{j}) = ab\sin\theta\,\hat{k}$$

$$\vec{a}\times(\vec{a}\times\vec{b}) = (ab\sin\theta)a(\hat{i}\times\hat{k}) = -a^2b\sin\theta\,\hat{j} \Rightarrow \vec{a}\times(\vec{b}\times\vec{a}) = a^2b\sin\theta\,\hat{j}$$

$$\Rightarrow \frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2} = b\sin\theta\,\hat{j}$$

Correct answer is (a)

2. The equation of the plane that is tangent to the surface $xyz = 8$ at the point (1, 2, 4) is **[CSIR Dec - 2011]**

(a) $x + 2y + 4z = 12$ (b) $4x + 2y + z = 12$
(c) $x + 4y + 2z = 12$ (d) $x + y + z = 7$

Soln. Suppose $T(x, y, z)$ be any point on the tangent plane of the given surface.

Since $\vec{\nabla}\phi$ is normal to the surface $\phi:\ xyz = 8$ at the point $P(1, 2, 4)$

Therefore, $\vec{\nabla}\phi$ is perpendicular to the vector $\overline{TP}$, lying in the tangent plane of the given surface.

Therefore, $\vec{\nabla}\phi\big|_{(1,2,4)}\cdot\overline{TP} = 0 \quad \Rightarrow 4x + 2y + z = 12$

Correct answer is (b)

3. A vector perpendicular to any vector that lies on the plane defined by $x + y + z = 5$, is **[CSIR June - 2012]**

(a) $\vec{i} + \vec{j}$ (b) $\vec{j} + \vec{k}$ (c) $\vec{i} + \vec{j} + \vec{k}$ (d) $2\vec{i} + 3\vec{j} + 5\vec{k}$

Soln. A vector perpendicular to any vector that lies on the plane defined by $\phi : x + y + z = 5$, is $\vec{\nabla}\phi = \hat{i} + \hat{j} + \hat{k}$

Correct answer is (c)

4. The unit normal vector at the point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ on the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is **[CSIR Dec - 2012]**

(a) $\dfrac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{a^2+b^2+c^2}}$ (b) $\dfrac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2+b^2+c^2}}$ (c) $\dfrac{b\hat{i} + c\hat{j} + a\hat{k}}{\sqrt{a^2+b^2+c^2}}$ (d) $\dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

Soln. Normal to the surface at point $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$:

$$\vec{\nabla}\phi\Big|_{\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)} = \frac{2x}{a^2}\hat{i} + \frac{2y}{b^2}\hat{j} + \frac{2z}{c^2}\hat{k} = \frac{2}{\sqrt{3}\,a}\hat{i} + \frac{2}{\sqrt{3}\,b}\hat{j} + \frac{2}{\sqrt{3}\,c}\hat{k}$$

Unit normal vector to the surface at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$:

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\frac{2}{\sqrt{3}}\left(\frac{1}{a}\hat{i} + \frac{1}{b}\hat{j} + \frac{1}{c}\hat{k}\right)}{\frac{2}{\sqrt{3}}\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$

Option (a) is correct but it is wrongly printed in the question.

5. A unit vector $\hat{n}$ on the xy-plane is at an angle of 120° with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60° if **[CSIR June - 2013]**

(a) $b = \sqrt{3}\,a/2$ (b) $b = 2a/\sqrt{3}$ (c) $b = a/2$ (d) $b = a$

Soln. Since, the unit vector makes an angle 120^0 with x-axis, then it makes an angle 30^0 with y-axis.

Therefore, $\hat{n} = \cos 120^0\,\hat{i} + \cos 30^0\,\hat{j} = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$

So, $\vec{u}$ and $\vec{v}$ can be wriitten as, $\vec{u} = \left(a - \frac{b}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}b\hat{j}$ and $\vec{v} = \left(b - \frac{a}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}a\hat{j}$

Since, the angle between $\vec{u}$ and $\vec{v}$ is 60^0, therefore $\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos 60^0$

$$\Rightarrow \left(a - \frac{b}{2}\right)\left(b - \frac{a}{2}\right) + \frac{3}{4}ab = \sqrt{\left(a - \frac{b}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2}\cdot\sqrt{\left(b - \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2}\,.\cos 60^0$$

$$\Rightarrow 2ab-\frac{b^2}{2}-\frac{a^2}{2}=\left(a^2+b^2-ab\right).\frac{1}{2} \quad \Rightarrow 4ab-b^2-a^2=a^2+b^2-ab$$

$$\Rightarrow 2a^2-5ab+2b^2=0 \quad \Rightarrow (2a-b)(2b-a)=0 \quad \Rightarrow a=\frac{b}{2} \text{ or } b=\frac{a}{2}$$

Correct answer is (c)

6. If $\vec{A}=yz\,\hat{i}+xz\,\hat{j}+xy\,\hat{k}$, then the integral $\oint_C \vec{A}.d\vec{l}$ (where C is along the perimeter of a rectangular area bounded by x = 0, x = a and y = 0, y = b) is **[CSIR Dec - 2013]**

(a) $\frac{1}{2}\left(a^3+b^3\right)$ (b) $\pi\left(ab^2+a^2b\right)$ (c) $\pi\left(a^3+b^3\right)$ (d) 0

Soln. $\vec{A}=yz\hat{i}+xz\hat{j}+xy\hat{k}$ $\Rightarrow$ $\vec{\nabla}\times\vec{A}=0$ i.e. field is conservative.

Since, line integral of a conservative field along a closed path is zero, then $\oint_C \vec{A}.\overrightarrow{d\ell}=0$

Correct answer is (d)

7. If $\vec{A}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ and C is the circle of unit radiius in the plane defined by z = 1, with the centre on the z-axis, then the value of the integral $\oint_C \vec{A}.\vec{d\ell}$ is **[CSIR June - 2014]**

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{4}$ (d) 0

Soln. $\vec{A}=yz\hat{i}+xz\hat{j}+xy\hat{k}$ $\Rightarrow$ $\vec{\nabla}\times\vec{A}=0$ i.e. field is conservative.

Since, line integral of a conservative field along a closed path is zero, then $\oint_C \vec{A}.\overrightarrow{d\ell}=0$

Correct answer is (d)

8. Let $\vec{r}$ denote the position vector of any point in three-dimensional space, and $r=|\vec{r}|$. Then

(a) $\vec{\nabla}\cdot\vec{r}=0$ and $\vec{\nabla}\times\vec{r}=\vec{r}/r$ (b) $\vec{\nabla}\cdot\vec{r}=0$ and $\nabla^2 r=0$

(c) $\vec{\nabla}\cdot\vec{r}=3$ and $\nabla^2\vec{r}=\vec{r}/r^2$ (d) $\vec{\nabla}\cdot\vec{r}=3$ and $\vec{\nabla}\times\vec{r}=0$

[CSIR Dec - 2014]

Soln. Given: Position vector of any point in three dimensional space is $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Therefore, $\vec{\nabla}.\vec{r}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=3$

and $$\vec{\nabla}\times\vec{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0$$

Correct Option is (d)

GATE Previous Years Questions with Solutions

1. If S is the closed surface enclosing a volume V and $\hat{n}$ is the unit normal vector to the surface and $\vec{r}$ is the positive vector, then the value of the folllowing integral $\iint_S \vec{r}\cdot\hat{n}dS$ is **[GATE 2001]**

(a) V (b) 2V (c) 0 (d) 3V

Soln. $\iint_S \vec{r}\cdot\hat{n}dS=\iiint_V\left(\vec{\nabla}\cdot\vec{r}\right)dV$ (using Divergence Theorem)

$$=3\iiint_V dV=3V$$

Correct option is (d)

2. Consider the set of vectors $\frac{1}{\sqrt{2}}(1,1,0),\frac{1}{\sqrt{2}}(0,1,1)$ and $\frac{1}{\sqrt{2}}(1,0,1)$ **[GATE 2001]**

(a) the three vectors are orthonormal

(b) the three vectors are linearly independent

(c) the three vectors cannot form a basis in a three-dimensional real vector space

(d) $\frac{1}{\sqrt{2}}(1,1,0)$ can be written as the linear combination of $\frac{1}{\sqrt{2}}(0,1,1)$ and $\frac{1}{\sqrt{2}}(1,0,1)$

Soln. $A=\frac{1}{\sqrt{2}}(1,1,0);\ B=\frac{1}{\sqrt{2}}(0,1,1);\ C=\frac{1}{\sqrt{2}}(1,0,1)$

$$\begin{vmatrix}A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\\ A_3 & B_3 & C_3\end{vmatrix}=\begin{vmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{vmatrix}=\frac{1}{\sqrt{2}}\left(\frac{1}{2}-0\right)-0\left(\frac{1}{2}-0\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{2}-0\right)=\frac{1}{\sqrt{2}}\neq 0$$

So, $\overline{A},\overline{B},\overline{C}$ are linearly independent.

Correct option is (b)

3. If $\vec{A}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z$, then $\nabla^2\vec{A}$ **[GATE 2001]**

(a) 1 (b) 3 (c) 0 (d) –3

Soln. $\vec{A}=x\hat{e}_x+y\hat{e}_y+z\hat{e}_z$

$$\nabla^2\vec{A}=\frac{\partial^2}{\partial x^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)+\frac{\partial^2}{\partial y^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)+\frac{\partial^2}{\partial z^2}\left(x\hat{e}_x+y\hat{e}_y+z\hat{e}_z\right)=0$$

Correct option is (c)

4. Which of the following vectors is orthogonal to the vector $\left(a\hat{i}+b\hat{j}\right)$, where a and b $(a \neq b)$ are constants, and $\hat{i}$ and $\hat{j}$ are unit orthogonal vectors? **[GATE 2002]**

(a) $-b\hat{i}+a\hat{j}$ (b) $-a\hat{i}+b\hat{j}$ (c) $-a\hat{i}-b\hat{j}$ (d) $-b\hat{i}-a\hat{j}$

Soln. $\left(a\hat{i}+b\hat{j}\right).\left(-b\hat{i}+a\hat{j}\right)=-ab+ab=0$

Correct option is (a)

5. A vector $\vec{A}=(5x+2y)\hat{i}+(3y-z)\hat{j}+(2x-az)\hat{k}$ is solenoidal if the constant a has a value.

(a) 4 (b) –4 (c) 8 (d) –8 **[GATE 2002]**

Soln. $\vec{A}$ will solenoidal if $\vec{\nabla}\cdot\vec{A}=0 \Rightarrow 5+3-a=0 \Rightarrow a=8$

Correct option is (c)

6. The unit vector normal to the surface $3x^2+4y=z$ at the ponit (1, 1, 7) is: **[GATE 2002]**

(a) $\left(-6\hat{\mathrm{i}}+4\hat{\mathrm{j}}+\hat{\mathrm{k}}\right)/\sqrt{53}$ (b) $\left(4\hat{\mathrm{i}}+6\hat{\mathrm{j}}-\hat{\mathrm{k}}\right)/\sqrt{53}$

(c) $\left(6\hat{\mathrm{i}}+4\hat{\mathrm{j}}-\hat{\mathrm{k}}\right)/\sqrt{53}$ (d) $\left(4\hat{\mathrm{i}}+6\hat{\mathrm{j}}+\hat{\mathrm{k}}\right)/\sqrt{53}$

Soln. Unit vector normal to the surface $3x^2+4y=z$ at the ponit (1, 1, 7) is

$$\hat{n}=\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}=\left.\frac{\left(6x\,\hat{i}+4\hat{j}-\hat{k}\right)}{\sqrt{36x^2+16+1}}\right|_{(1,1,7)}=\frac{\left(6\,\hat{i}+4\hat{j}-\hat{k}\right)}{\sqrt{53}}$$

Correct option is (c)

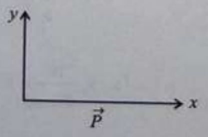
7. The two vectors $\vec{P}=\hat{i},\vec{q}=\left(\hat{i}+\hat{j}\right)/\sqrt{2}$ are **[GATE 2003]**

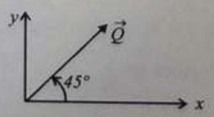
(a) related by a rotation (b) related by a reflection through the xy-plane

(c) related by an inversion (d) not linearly independent

Soln. $\vec{P}=\hat{i}=\cos 0°\,\hat{i}+\cos 90°\,\hat{j}$

$\vec{Q}=\left(\hat{i}+\hat{j}\right)/\sqrt{2}=\cos 45°\,\hat{i}+\sin 45°\,\hat{j}$





Correct option is (c)

8. The curl of the vector $\vec{A}=z\hat{i}+x\hat{j}+y\hat{k}$ is given by **[GATE 2003]**

(a) $\hat{i}+\hat{j}+\hat{k}$ (b) $\hat{i}-\hat{j}+\hat{k}$ (c) $\hat{i}+\hat{j}-\hat{k}$ (d) $-\hat{i}-\hat{j}-\hat{k}$

Soln. $$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ z & x & y\end{vmatrix}=\left(\hat{i}+\hat{j}+\hat{k}\right)$$

Correct option is (a)

Common data for Q. 9 and Q.10

Consider the vector $\vec{V}=\dfrac{\vec{r}}{r^3}$. **[GATE 2003]**

9. The surface integral of this vector over the surface of a cube of size and centred at the origin

(a) 0 (b) 2π (c) $2\pi a^3$ (d) 4π

Soln. $\iint\limits_S \vec{V}\cdot\hat{n}ds=\iiint\limits_v\left(\vec{\nabla}\cdot\vec{V}\right)dv$ (Divergence theorem)

$$=\iiint\limits_v\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)dv=4\pi\iiint\limits_v\delta^3\left(\vec{r}\right)dv$$

$$=4\pi\iiint\limits_v\delta(x)\delta(y)\delta(z)dv=4\pi$$

Correct option is (d)

10. Which one of the following is NOT correct?

(a) Value of the line integral of this vector around any closed curve is zero

(b) This vector can be written as the gradient of some scalar function

(c) The line integral of this vector from point P to point Q is independent of the path taken

(d) This vector can represent the magnetic field of some current distribution.

Soln. $\vec{\nabla}\times\left(\dfrac{\vec{r}}{r^3}\right)=0$

Therefore, $\vec{V}=\dfrac{\vec{r}}{r^3}$ is a conservative field.

So, $\oint\limits_C\vec{V}\cdot d\vec{r}=0$ and $\vec{V}=\vec{\nabla}\phi$ and $\int_P^Q\vec{V}\cdot d\vec{r}=$ independent of path

But, magnetic field $\vec{B}$ is not conservative.

Correct option is (d)

11. For the function $\phi=x^2y+xy$ the value of $\left|\vec{\nabla}\phi\right|$ at $x=y=1$ is **[GATE 2004]**

(a) 5 (b) $\sqrt{5}$ (c) 13 (d) $\sqrt{13}$

Soln. $\phi=x^2y+xy \Rightarrow \vec{\nabla}\phi=(2xy+y)\hat{i}+\left(x^2+x\right)\hat{j}$

$\left.\vec{\nabla}\phi\right|_{(1,1,1)}=3\hat{i}+2\hat{j}\,;\left|\vec{\nabla}\phi\right|_{(1,1,1)}=\sqrt{13}$

Correct option is (d)

12. The unit normal to the curve $x^3y^2+xy=17$ at the point $(2,0)$ is **[GATE 2005]**

(a) $\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$ (b) $-\hat{i}$ (c) $-\hat{j}$ (d) $\hat{j}$

Soln. Unit normal vector to the curve : $\phi=x^3y^2+xy=17$

$$\hat{n}=\vec{\nabla}\phi/\left|\vec{\nabla}\phi\right|\Big|_{(2,0)}=\frac{\left(3x^2y^2+y\right)+\left(2x^3y+x\right)\hat{j}}{\left|\vec{\nabla}\phi\right|}\Bigg|_{(2,0)}=\hat{j}$$

Correct option is (d)

13. If a vector field $\vec{F}=x\hat{i}+2y\hat{j}+3z\hat{k}$, then $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right)$ is **[GATE 2005]**

(a) 0 (b) $\hat{i}$ (c) $2\hat{j}$ (d) $3\hat{k}$

Soln. $\vec{F}=x\hat{i}+2y\hat{j}+3z\hat{k}$

$\vec{\nabla}\times\vec{F}=0\ \Rightarrow\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right)=0$

Correct option is (a)

14. Given the four vectors, $u_1=\begin{pmatrix}1\\2\\1\end{pmatrix},u_2=\begin{pmatrix}3\\-5\\1\end{pmatrix},u_3=\begin{pmatrix}2\\4\\-8\end{pmatrix},u_4=\begin{pmatrix}3\\6\\-12\end{pmatrix}$, the linearly dependent6 pair is

(a) u_1,u_2 (b) u_1,u_3 (c) u_1,u_4 (d) u_3,u_4 **[GATE 2005]**

Soln. $u_4=\frac{3}{2}u_3$ i.e. u_3,u_4 are linearly dependent

Correct option is (d)

15. The value of $\oint\limits_S\frac{\vec{r}.d\vec{s}}{r^3}$, where $\vec{r}$ is the position vector ans S is the closed surface enclosing the origin, is **[GATE 2006]**

(a) zero (b) π (c) 4π (d) 8π

Soln. $\oint\limits_s\frac{\vec{r}\cdot d\vec{s}}{r^3}=\int\limits_V\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)dV$ (using divergence theorem)

Since the closed surface encloses the origin i.e. $\vec{r}=0$

Therefore, $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)=4\pi b^3\left(\vec{r}\right)$

So, $\int\limits_V\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)dV=4\pi\int\limits_V\delta^3\left(\vec{r}\right)dV=4\pi$

Correct option is (c)

16. If $\vec{r}=x\hat{i}+y\hat{j}$, then

(a) $\vec{\nabla}\cdot\vec{r}=0$ and $\vec{\nabla}\left|\vec{r}\right|=\vec{r}$ (b) $\vec{\nabla}\cdot\vec{r}=2$ and $\vec{\nabla}\left|\vec{r}\right|=\vec{r}$

(c) $\vec{\nabla}\cdot\vec{r}=2$ and $\vec{\nabla}\left|\vec{r}\right|=\frac{\vec{r}}{r}$ (d) $\vec{\nabla}\cdot\vec{r}=3$ and $\vec{\nabla}\left|\vec{r}\right|=\frac{\vec{r}}{r}$ **[GATE 2007]**

Soln. $\vec{r}=x\hat{i}+y\hat{j}\ \Rightarrow\vec{\nabla}\cdot\vec{r}=2$

$\vec{\nabla}\left|\vec{r}\right|=\vec{\nabla}r=1\cdot\hat{r}=\frac{\vec{r}}{\left|\vec{r}\right|}$

Correct option is (c)

17. The curl of the vector field is $\vec{F}$ is $2\hat{x}$. Identify the appropriate vector field $\vec{F}$ from the choices given below: **[GATE 2008]**

(a) $\vec{F}=2z\hat{x}+3z\hat{y}+5y\hat{z}$ (b) $\vec{F}=3z\hat{x}+5y\hat{z}$

(c) $\vec{F}=3x\hat{y}+5y\hat{z}$ (d) $\vec{F}=2x\hat{y}+5y\hat{z}$

Soln. $\vec{F}=3z\hat{y}+5y\hat{z}$

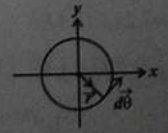
$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\0&3z&5y\end{vmatrix}=2\hat{i}$$

Correct option is (b)

18. The value of contour integral $\left|\int\limits_C\vec{r}\times d\vec{\theta}\right|$, for a circle C of radius r with centre at the origin is

(a) $2\pi r$ (b) $\frac{r^2}{2}$ (c) πr^2 (d) r **[GATE 2009]**

Soln. $\left|\int\limits_C\vec{r}\times d\vec{\theta}\right|=\int\limits_C rd\theta\sin 90^\circ=r\int_0^{2\pi}d\theta=2\pi r$



Correct option is (a)

19. Consider the set of vectors in three-dimensional real vector space R^3, $S=\{(1,1,1),(1,-1,1),(1,1,-1)\}$ **[GATE 2009]**

Which of the following statement is true?

(a) S is not a linearly independent set

(b) S is a basis for R^3

(c) The vectors in S are orthogonal

(d) An orthogonal set of vectors cannot be generated from S

Soln. $\begin{vmatrix}1&1&1\\1&-1&1\\1&1&-1\end{vmatrix}=1(1-1)-1(-1-1)+1(1+1)=4\neq0$

Correct option is (b)

20. If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that **[GATE 2011]**

(a) $\vec{\nabla}\times\vec{F}=0$ (b) $\vec{\nabla}\cdot\vec{F}=0$ (c) $\vec{\nabla}V=0$ (d) $\nabla^2V=0$

Soln. If force $\vec{F}$ is derivable from a potential function $V(r)$, then it will be conservative in nature i.e.

$\vec{\nabla}\times\vec{F}=0$

Correct option is (a)

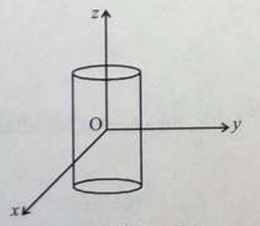
21. The unit vector normal to the surface $x^2+y^2-z=1$ at the point $P(1, 1, 1)$ is : **[GATE-2011]**

(a) $\dfrac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ (b) $\dfrac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$ (c) $\dfrac{\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$ (d) $\dfrac{2\hat{i}+2\hat{j}-\hat{k}}{3}$

Soln. Unit vector normal to the surface $\hat{n}=\left.\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\right|_{(1,1,1)}=\dfrac{2\hat{i}+2\hat{j}-\hat{k}}{3}$

Correct option is (d)

22. Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r}=\hat{i}x+\hat{j}y+\hat{k}z$ be the position vector and $\hat{n}$ a unit vector normal to the surface. The surface integral $\int_S \vec{r}\cdot\hat{n}\,dS$ over the closed surface of the cylinder is : **[GATE-2011]**



(a) $2\pi a^2(a+h)$ (b) $3\pi a^2h$ (c) $2\pi a^2h$ (d) zero

Soln. $\int_S \vec{r}\cdot\hat{n}\,dS=\int_V(\vec{\nabla}\cdot\vec{r})\,dV$ (Using divergence theorem)

$$=3\int_V dV=3V=3\pi a^2h$$

Correct option is (b)

23. Identify the correct statement for the following vectors $\vec{a}=3\hat{i}+2\hat{j}$ and $\vec{b}=\hat{i}+2\hat{i}$ **[GATE-2012]**

(a) The vectors $\vec{a}$ and $\vec{b}$ are linearly independent
(b) The vectors $\vec{a}$ and $\vec{b}$ are linearly dependent
(c) The vectors $\vec{a}$ and $\vec{b}$ are orthogonal
(d) The vectors $\vec{a}$ and $\vec{b}$ are normalized

Soln. $\begin{vmatrix}3 & 1\\ 2 & 2\end{vmatrix}=4\neq 0$; $\vec{a}$ and $\vec{b}$ linearly independent.

Correct option is (a)

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24. For a sclar function ϕ satisfying the laplace equation, $\nabla\phi$ has **[GATE-2013]**

(a) zero curl and non-zero divergence (b) non-zero curl and zero divergence
(c) zero curl and zero divergence (d) mon-zero curl and non-zero divergence

Soln. $\vec{\nabla}\times\vec{\nabla}\phi=0$

From Laplacian equation $\nabla^2\phi=0\Rightarrow\vec{\nabla}\cdot(\vec{\nabla}\phi)=0$
Correct option is (c)

25. If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}\left[\vec{A}.\left(\vec{B}\times\vec{r}\right)\right]$ is **[GATE-2013]**

(a) $\vec{A}.\vec{B}$ (b) $\vec{A}\times\vec{B}$ (c) $\vec{r}$ (d) zero

Soln. $\vec{\nabla}\left[\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right]=\vec{\nabla}\left[\vec{r}\cdot\left(\vec{A}\times\vec{B}\right)\right]=\vec{\nabla}\left[\left(\vec{A}\times\vec{B}\right)\cdot\vec{r}\right]$

If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\left(\vec{A}\times\vec{B}\right)$ is also constant vector. For constant vector $\vec{a}$

$$\vec{\nabla}\left(\vec{a}\cdot\vec{r}\right)=\vec{a}$$

Therefore, $\vec{\nabla}\left[\left(\vec{A}\times\vec{B}\right)\cdot\vec{r}\right]=\left(\vec{A}\times\vec{B}\right)$

Correct option is (b)

26. The unit vector perpendicular to the surface $x^2+y^2+z^2=3$ at the point $(1, 1, 1)$ is : **[GATE-2014]**

(a) $\dfrac{\hat{x}+\hat{y}-\hat{z}}{\sqrt{3}}$ (b) $\dfrac{\hat{x}-\hat{y}-\hat{z}}{\sqrt{3}}$ (c) $\dfrac{\hat{x}-\hat{y}+\hat{z}}{\sqrt{3}}$ (d) $\dfrac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}$

Soln. Unit vector perpendicular to the surface

$$\hat{n}=\left.\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\right|_{(1,1,1)}=\left.\frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}\right|_{(1,1,1)}=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

Correct option is (d)

27. Four forces are given below in Cartesian and spherical polar coordinates. **[GATE 2015]**

(i) $\vec{F}_1=K\exp\left(\dfrac{-r^2}{R^2}\right)\hat{r}$ (ii) $\vec{F}_2=K\left(x^3\hat{y}-y^3\hat{z}\right)$

(iii) $\vec{F}_3=K\left(x^3\hat{x}+y^3\hat{y}\right)$ (iv) $\vec{F}_4=K\left(\dfrac{\hat{\phi}}{r}\right)$

where K is a constant. Identify the correct option.
(a) (iii) and (iv) are conservative but (i) and (ii) are not
(b) (i) and (ii) are conservative but (iii) and (iv) are not
(c) (ii) and (iii) are conservative but (i) and (iv) are not
(d) (i) and (iii) are conservative but (ii) and (iv) are not

Soln. $$\vec{\nabla}\times\vec{F}_1=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ k.\exp\left(-r^2/R^2\right) & 0 & 0\end{vmatrix}=0$$

$$\vec{\nabla}\times\vec{F}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & kx^3 & -kz^3\end{vmatrix}=3kx^2\hat{k}$$

$$\vec{\nabla}\times\vec{F}_3=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ kx^3 & ky^3 & 0\end{vmatrix}=0$$

$$\vec{\nabla}\times\vec{F}_4=\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 0 & 0 & \frac{k}{r}\end{vmatrix}=\frac{1}{r^2\sin\theta}r\hat{\theta}\left[-\frac{k}{r^2}\right]=-\frac{k}{r^3\sin\theta}\hat{\theta}$$

Correct option is (d)

IIT-JAM Previous Year Questions with Solutions

1. The work done by a force in moving particle of mass 'm' from any point (x, y) to a neighbouring point $(x+dx,\ y+dy)$ is given by $dW=2xydx+x^2dy$. The work done for a complete cycle around a unit circle is:

(a) 0 (b) 1 (c) 3 (d) 2π

Soln. Work done for a complete cycle around a unit circle is

$$W=\iint dW=\oint_C\left[2xydx+x^2dy\right]=\oint_C\left[2xy\hat{i}+x^2\hat{j}\right].d\vec{r}$$

$$=\iint_S\left[\vec{\nabla}\times\left(2xy\hat{i}+x^2\hat{j}\right)\right].d\vec{s}=0$$

Correct answer is (a)

2. The equation of a surface of revolution is $z=\pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2}$. The unit normal to the surface at the point $A\left(\sqrt{\frac{3}{2}},0,1\right)$ is:

(a) $\sqrt{\frac{3}{5}}i+\frac{2}{\sqrt{10}}\hat{k}$ (b) $\sqrt{\frac{3}{5}}i-\frac{2}{\sqrt{10}}\hat{k}$ (c) $\sqrt{\frac{3}{5}}i+\frac{2}{\sqrt{5}}\hat{k}$ (d) $\sqrt{\frac{3}{10}}i+\frac{2}{\sqrt{10}}\hat{j}$

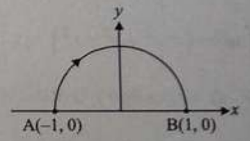
Soln. Equation of surface : $z=\pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2}$ $\Rightarrow 3x^2+3y^2-2x^2=0$

Unit normal to the surface at point $A\left(\sqrt{\frac{2}{3}},0,1\right)$

$$\hat{n}=\vec{\nabla}\phi/\left|\vec{\nabla}\phi\right|=\left.\frac{6x\hat{i}+6y\hat{j}-4z\hat{k}}{\sqrt{36x^2+36y^2+16z^2}}\right|_{\left(\sqrt{\frac{2}{3}},0,1\right)}=\sqrt{\frac{3}{5}}\hat{i}-\frac{2}{\sqrt{10}}\hat{k}$$

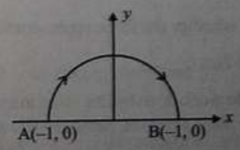
Correct answer is (b)

3. The line integral $\int_A^B\vec{F}.\overline{dl}$, where $\vec{F}=\frac{x}{\sqrt{x^2+y^2}}\hat{x}+\frac{y}{\sqrt{x^2+y^2}}\hat{y}$, along the semi-circular path as shown in the figure below is:



(a) – 2 (b) 0 (c) 2 (d) 4

Soln. $$\int_A^B\vec{F}.d\vec{\ell}=\int_{(-1,0)}^{(1,0)}\frac{xdx+ydy}{\sqrt{x^2+y^2}}$$



Putting $x=\cos\theta\Rightarrow dx=-\sin\theta d\theta$

$y=\sin\theta\Rightarrow dy=\cos\theta d\theta$

$$I=\int_\pi^0\frac{\left[-\sin\theta\cos\theta+\sin\theta\cos\theta\right]d\theta}{1}=0$$

Correct answer is (b)

4. If $\vec{F}$ is a constant vector and $\vec{r}$ is the position vector then $\vec{\nabla}\left(\vec{F}.\vec{r}\right)$ would be

(a) $\left(\vec{\nabla}.\vec{r}\right)\vec{F}$ (b) $\vec{F}$ (c) $\left(\vec{\nabla}.\vec{F}\right)\vec{r}$ (d) $\left|\vec{r}\right|\vec{F}$

Soln. $\vec{F}$ is constant vector i.e. $\vec{F}=\vec{F}_1\hat{i}+\vec{F}_2\hat{j}+\vec{F}_3\hat{k}$ (say)

$$\vec{\nabla}\left(\vec{F}.\vec{r}\right)=\vec{\nabla}\left(F_1x+F_2y+F_3z\right)=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}=\vec{F}$$

Correct answer is (b)

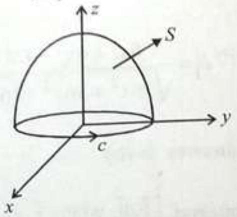
5. (a) Consider a constant vector field $\vec{\upsilon}=\upsilon_0\hat{k}$. Find any one of the many possibe vectors $\vec{u}$, for which

$\vec{\nabla}\times\vec{u}=\vec{v}$

(b) Using Stoke's theorem, evaluate the flux associated with the field $\vec{v}$ through the curved hemispherical surface defined by $x^2+y^2+z^2=r^2, z>0$.

Soln. (a) Suppose $\vec{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$

Given : $\vec{\nabla}\times\vec{u}=\vec{v} \Rightarrow \left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right)\hat{i}-\left(\frac{\partial u_z}{\partial x}-\frac{\partial u_x}{\partial z}\right)\hat{j}+\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)=\hat{k}=v_0\hat{k}$



So, $\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)=v_0$

There are many possible solutions:

(i) $u_x=0, v_y=v_0 x\, u_z=0 \qquad \Rightarrow \vec{u}=v_0 x\hat{j}$

(ii) $u_x=-v_0 y, v_y=0,\ u_z=0 \qquad \Rightarrow \vec{u}=-v_0 y\hat{i}$ and so on.

(b) Flux associated with the field $\vec{v}=\int\limits_s \vec{v}.\hat{n}ds=\int\limits_s\left(\vec{\nabla}\times\vec{u}\right).\hat{n}ds$

(Using Stokes theorem)

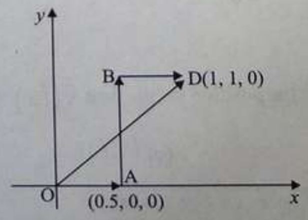
$=\oint\limits_c \vec{u}.d\vec{r}\ \left(\text{taking } \vec{u}=v_0 x\hat{j}\right)=v_0\int\limits_c xdy=\left(c:x^2+y^2=r^2\right)$

(Using $x=r\cos\theta$ and $y=r\sin\theta$, $dy=r\cos\theta d\theta$)

$$\Rightarrow \quad \int\limits_S \vec{v}.\hat{n}ds=v_0\int\limits_{\theta=0}^{2\pi} r^2\cos^2\theta d\theta=\pi r^2 v_0$$

6. (a) Determine whether the force represented by $\vec{F}(x,y)=k\left[\left(x^2+y^2\right)\hat{i}+2xy\,\hat{j}\right]$ is conservative or not. Here $k = 1\ Nm^{-2}$.

(b) Calculate the work done by this force in moving a particle from the origin O(0, 0, 0) to the point D(1, 1, 0) on the $z=0$ plane along the paths OABD and OD as shown in figure, where the coordinates are measured in metres.



Soln. (a) $\vec{F}(x,y)=k\left[\left(x^2+y^2\right)\hat{i}+2xy\hat{j}\right]$

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+y^2 & 2xy & 0\end{vmatrix}=\hat{k}(2y-2y)=0$$

(b) Since, $\vec{\nabla}\times\vec{F}=0$ i.e. force $\vec{F}$ is conservative

Therefore, work done by a conservative field in moving a particle from 0 to D, is independent of path i.e.

$$W_{OBCD}=W_{OD}=\phi(D)-\phi(D)$$

where ϕ is scalar potential corresponding to force field $\vec{F}$

Calculation of $\phi:\vec{F}=\vec{\nabla}\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$

So, $\frac{\partial\phi}{\partial x}=k\left(x^2+y^2\right) \Rightarrow \phi=k\int\limits_{y=\text{constant}}\left(x^2+y^2\right)dx=k\left[\frac{x^3}{3}+xy^2\right]+f_1(y)$

$\frac{\partial\phi}{\partial x}=k(2xy) \Rightarrow \phi=k\int\limits_{x=\text{constant}} 2xydy=kxy^2+f_2(y)$

Therefore, $(x,y,z)=k\left[\frac{x^3}{3}+xy^2\right]+c$

Work done in moving the particle from 0 to D

$$W_{OD}=k\left[\frac{x^3}{3}+xy^2\right]_{(0,0,0)}^{(1,1,0)} \quad\Rightarrow\quad W_{OD}=\frac{4}{3}k \text{ units.}$$

7. For the vector field $\vec{V}=xz^2\hat{i}-yz^2\hat{j}+z\left(x^2-y^2\right)\hat{k}$

(a) Calculate the volume integral of the divergence of $\vec{V}$ over the region defined by $-a\le x\le a,\ -b\le y\le b$ and $0\le z\le c$.

(b) Calculate the flux of $\vec{V}$ out of region through the surface at $z=c$. Hence deduce the net flux through the rest of the boundary of the region.

Soln. (a) $\vec{V}=xz^2\hat{i}-yz^2\hat{j}+z\left(x^2-y^2\right)\hat{k}$

$$\iiint\limits_v\left(\vec{\nabla}.\vec{V}\right)=\int\limits_0^c\int\limits_{-b}^b\int\limits_{-a}^a\left(x^2-y^2\right)dxdydz=\int\limits_{-a}^a x^2dx\int\limits_{-b}^b dy\int\limits_0^c dz-\int\limits_{-a}^a dx\int\limits_{-b}^b y^2dy\int\limits_0^x dz=\frac{4}{3}abc\left(a^2-b^2\right)$$

(b) Flux of $\vec{V}$ out the region through the surface $z=c$

$=\int\vec{V}.\hat{n}ds=\iint z\left(x^2-y^2\right)dxdy \qquad \left[\text{for } z=c \text{ surface } \hat{n}=\hat{k}, ds=dxdy\right]$

$=c\int_{-b}^{b}\int_{-a}^{a}\left(x^2-y^2\right)dxdy=\frac{4}{3}abc\left(a^2-b^2\right)$

Flux of the vector field $\vec{V}$ across the closed surface of the region defined by $-a \le x \le a, -b \le y \le b, 0 \le z \le c$

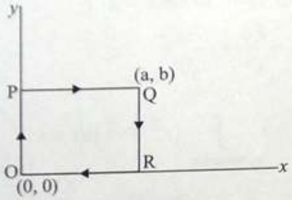
$$= \oiint_s \vec{V}.\hat{n}ds = \iiint_v \left(\vec{\nabla}.\vec{V}\right) dv = \frac{4}{3}abc\left(a^2 - b^2\right)$$

So, flux through the rest of the boundary region = 0

8. How many work is done when an object from $O \rightarrow P \rightarrow Q \rightarrow R \rightarrow O$ in a force field given by

$$F(x,y) = \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j}$$

along the rectangular path shown. Find the answer by evaluating the line integral and also by using the Stoke's theorem.



Soln. Work done by the force field $\vec{F} = \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j}$

In moving a particle along $O \rightarrow P \rightarrow Q \rightarrow R \rightarrow O$ is equal to

$$\oint_C \vec{F}.d\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F}\right).\hat{n}ds$$

C : path $O \rightarrow P \rightarrow Q \rightarrow R \rightarrow O$

S : rectangular sheet OPQR in x-y plane.

$$\oint_C \vec{F}.d\vec{r} = \int_{OP} \vec{F}.d\vec{r} + \int_{PQ} \vec{F}.d\vec{r} + \int_{QR} \vec{F}.d\vec{r} + \int_{RO} \vec{F}.d\vec{r}$$

(i) For OP : x = 0, dx = 0, dy = 0 and x = 0 at b : $\int_{OP} \vec{F}.d\vec{r} = \int_{OP} 2xydy = 0$

(ii) For PQ : y = b, dy = 0 and x = 0 to a : $\int_{PQ} \vec{F}.d\vec{r} = \int_{PQ} \left(x^2 - y^2\right)dx = \int_0^a \left(x^2 - b^2\right)dx = \frac{a^3}{3} - ab^2$

(iii) For QR : x = a, dx = 0 and y = b to 0 : $\int_{QR} \vec{F}.d\vec{r} = \int_{QR} 2xydy = 0 = 2a\int_b^0 ydy = -ab^2$

(iv) For RO : y = 0, dy = 0 and x = a to 0 : $\int_{RO} \vec{F}.d\vec{r} = \int_{RO} \left(x^2 - y^2\right)dx = \int_a^0 x^2dx = -\frac{a^3}{3}$

$$\text{Work done} = \oint_C \vec{F}.d\vec{r} = 0 + \frac{a^3}{3} - ab^2 - ab^2 - \frac{a^3}{3} = -2ab^2$$

$$\iint_S \left(\vec{\nabla} \times \vec{F}\right).\hat{n}ds = \iint_S \left(4y\hat{k}\right).\left(-\hat{k}\right)dxdy = -4\int_0^b ydy\int_0^a sdx = -2ab^2$$

$\left[\hat{n} = -\hat{k}, dS = dxdy\right]$ → due to clockwise orientation of c.

Therefore, $\iint_S \left(\vec{\nabla} \times \vec{F}\right).\hat{n}ds = \oint_C \vec{F}.d\vec{r}$

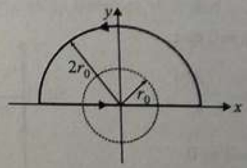
9. A vector field is given by,

$$\vec{F}(\vec{r}) = \begin{cases} \alpha\left(x\hat{j} - y\hat{i}\right) & \text{for } \left(x^2 + y^2\right) \le r_0^2 \text{ (region-I)} \\ \alpha r_0^2 \dfrac{\left(x\hat{j} - y\hat{i}\right)}{\left(x^2 + y^2\right)} & \text{for } \left(x^2 + y^2\right) > r_0^2 \text{ (region – II)} \end{cases}$$

Here α and r_0 are two constants.

(a) Find the curl of this field both the regions.

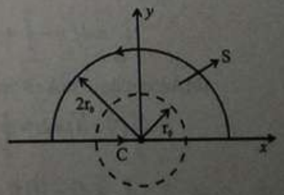
(b) Find the line integral $\oint \vec{F}.d\vec{\ell}$ along the closed semicircle path of radius $2r_0$ as shown in the figure below.



Soln. A vector field is defined as

$$\vec{F}(\vec{r}) = \alpha\left(x\hat{j} - y\hat{i}\right) \text{ for } x^2 + y^2 \le r_0^2$$

$$= \alpha r_0^2 \frac{\left(x\hat{j} - y\hat{i}\right)}{\left(x^2 + y^2\right)} \text{ for } x^2 + y^2 \ge r_0^2$$



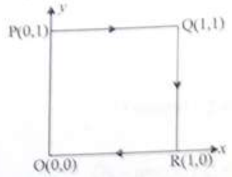
$$\text{(a) Region I : } \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\alpha y & \alpha x & 0 \end{vmatrix} = 2\alpha\hat{k}$$

$$\text{Region II : } \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\alpha r_0^2 y}{x^2 + y^2} & \frac{\alpha r_0^2 x}{x^2 + y^2} & 0 \end{vmatrix} = 0$$

$$\text{(b) } \oint_C \vec{F}.d\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F}\right).\hat{n}ds = \iint_I \left(\vec{\nabla} \times \vec{F}\right).\hat{n}ds + \iint_{II} \left(\vec{\nabla} \times \vec{F}\right).\hat{n}ds$$

$$= \iint_I 2\alpha\hat{k}.\hat{k}ds \quad \left[\text{for region I:} \hat{n} = \hat{k}\right] = 2\alpha\iint_I ds = 2\alpha.\frac{\pi r_0^2}{2} = \alpha\pi r_0^2$$

10. Consider a vector $\vec{A} = -4yx^2\hat{x} - 3y^2\hat{y}$



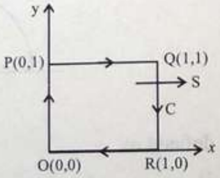
(a) Calculate the line integral $\int \vec{A}.\vec{d\ell}$ from point $P \to O$ along the path $P \to Q \to R \to O$ as shown in the figure.

(b) Using Stoke's theorem appropriately. Calculate $\int \vec{A}.\vec{d\ell}$ for the same path $P \to Q \to R \to O$.

Soln. $$\int\limits_{P\to O} \vec{A}.d\vec{\ell} = \int\limits_{P\to Q} \vec{A}.d\vec{\ell} + \int\limits_{Q\to R} \vec{A}.d\vec{\ell} + \int\limits_{R\to O} \vec{A}.d\vec{\ell}$$

(1) For $P \to Q: y = 1, ; dy = 0; x = 0$ to 1

$$\int\limits_{P\to Q} \vec{A}.d\vec{\ell} = -\int\limits_{x=0}^{1} 4yx^2 dx = -\frac{4}{3}$$



(2) For $Q \to R: x = 1, dx = 0; y = 1$ to 0

$$\int\limits_{Q\to R} \vec{A}.d\vec{\ell} = -\int\limits_{y=1}^{0} 3y^2 dy = 1$$

(3) For $R \to 0: y = 0, dy = 0; x = 1$ to 0

$$\int\limits_{R\to 0} \vec{A}.d\vec{\ell} = -\int\limits_{x=1}^{0} 4yx^2 dx = 0; \quad \int\limits_{P\to 0} \vec{A}.d\vec{\ell} = -\frac{4}{3} + 1 + 0 = -\frac{1}{3}$$

(b) According to stoke's theorem,

$$\iint\limits_S (\vec{\nabla}\times\vec{A}).\hat{n}ds = \oint\limits_C \vec{A}.d\vec{\ell} = \int\limits_{O\to P} \vec{A}.d\vec{\ell} + \int\limits_{P\to O} \vec{A}.d\vec{\ell}$$

$$\Rightarrow \quad \int\limits_{P\to O} \vec{A}.d\vec{\ell} = \iint\limits_S (\vec{\nabla}\times\vec{A}).\hat{n}ds - \int\limits_{O\to P} \vec{A}.d\vec{\ell}$$

$$\iint\limits_S (\vec{\nabla}\times\vec{A}).\hat{n}ds = \iint\limits_s (4x^2\hat{k}).(-\hat{k})dxdy = -\int_0^1\int_0^1 4x^2 dxdy = -\frac{4}{3}$$

[Here, $\hat{n} = -\hat{k}$ due to clockwise orientation of C and dS dxdy]

For $O \to P: x = 0, dx = 0; y = 0$ to 1

$$\int\limits_{O\to P} \vec{A}.d\vec{\ell} = -\int_0^1 3y^2 dy = -1$$

Therefore, $$\int\limits_{P\to O} \vec{A}.d\vec{\ell} = \iint\limits_S (\vec{\nabla}\times\vec{A}).\hat{n}ds - \int\limits_{O\to P} \vec{A}.d\vec{\ell} = -\frac{4}{3} + 1 = -\frac{1}{3}$$

11. For vectors $\vec{a} = \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$ and $\vec{c} = \hat{j} - \hat{k}$, the vector product $\vec{a}\times(\vec{b}\times\vec{c})$ is

(a) in the same direction as $\vec{c}$

(b) in the direction opposite to $\vec{c}$

(c) in the same direction as $\vec{b}$

(d) in the direction opposite to $\vec{b}$

Soln. $\vec{a}\times(\vec{b}\times\vec{c})$ [Using vector triple product]

$= \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b}) = 2\vec{c}$

Correct answer is (a)

12. If the surface integral of the field $\vec{A}(x, y, z) = 2\alpha x\hat{i} + \beta y\hat{j} - 3\gamma z\hat{k}$ over the closed surface of an arbitrary unit sphere is to be zero, then the relationship between α, β and γ is

(a) $\alpha + \beta/6 - \gamma = 0$

(b) $\alpha/3 + \beta/6 - \gamma/2 = 0$

(c) $\alpha/2 + \beta - \gamma/3 = 0$

(d) $2/\alpha + 1/\beta - 3/\gamma = 0$

Soln. According to Divergence Theorem,

$$\oiint\limits_s \vec{A}.d\vec{s} = \iiint\limits_V (\vec{\nabla}.\vec{A})dV$$

By questions, $$\oiint\limits_s \vec{A}.d\vec{s} = \iiint\limits_V (\vec{\nabla}.\vec{A})dV = 0$$

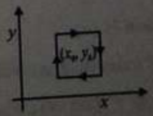
$$\Rightarrow \quad \iiint\limits_V (2\alpha + \beta - 3\gamma)dV = 0$$

$$\Rightarrow \quad (2\alpha + \beta - 3\gamma) = 0$$

$$\Rightarrow \quad \frac{\alpha}{3} + \frac{\beta}{6} - \frac{\gamma}{2} = 0$$

Correct answer is (b)

13. The line integral $\oint \vec{A}.d\vec{l}$ of a vector field $\vec{A}(x, y) = \frac{1}{r^2}(-y\hat{i} + x\hat{j})$, where $r^2 = x^2 + y^2$, is taken around a square (see figure) of side unit length and centered at (x_0, y_0) with $|x_0| > \frac{1}{2}$ and $|y_0| > \frac{1}{2}$. If the value of the integral is L, then



(a) L depends on (x_0, y_0)

(b) L is independent of (x_0, y_0) and its value is -1

(c) L is independent of (x_0, y_0) and its value is 0

(d) L is independent of (x_0, y_0) and its value is 2

Soln. $\vec{A} = \frac{1}{r^2}(-y\hat{i} + x\hat{j}) = \frac{1}{x^2 + y^2}(-y\hat{i} + x\hat{j})$

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0\end{vmatrix}=0$$

and the field in defined everwhere within and on the square of side unit length and centered at (x_0, y_0) with $|x_0|>\frac{1}{2}$ and $|y_0|>\frac{1}{2}$

Therefore, $L=\oint_c \vec{A}.d\vec{\ell}=\iint_s(\vec{\nabla}\times\vec{A}).d\vec{s}$ (Stokes Theorem)

$= 0$ (independent of $[x_0, y_0]$)

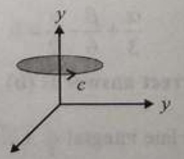
Correct answer is (c)

14. Consider a vector field $\vec{F}=y\hat{i}+xz^3\hat{j}-zy\hat{k}$. Let C be the circle $x^2+y^2=4$ on the plane $z=2$, oriented counter-clockwise. The value of contour integral $\oint_C \vec{F}\cdot d\vec{r}$ is

(a) 28π (b) 4π (c) -4π (d) -28π

Soln. $\vec{F}=y\hat{i}+xz^3\hat{j}-yz\hat{k}$

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & xz^3 & -yz\end{vmatrix}=\hat{i}(-z-3xz^2)-\hat{j}(0-0)+\hat{k}(z^3-1)$$



Now, using stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r}=\iint_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S}=\iint(z^3-1)\,dxdy$$

[Since, circle is in z = 2 plane] $d\vec{S}=dxdy\hat{k}$

$$=7\iint dxdy=7\times\pi(2)^2=28\pi \qquad c: x^2+y^2=4, z=2$$

Correct option is (a)

15. A particle of mass m is moving in x-y plane. At any given time t, its position vector is given by $\vec{r}(t)=A\cos\omega t\hat{i}+B\sin\omega t\hat{j}$, where A, B and ω are contants with $A\neq B$. Which of the following statements are true?

(a) orbit of the particle is an ellipse
(b) speed of the particle is constant

(c) at any given time t, the particle experiences a force towards origin
(d) the angular momentum of the particle is $m\omega AB\hat{k}$

Soln. $\vec{r}=A\cos\omega t\,\hat{i}+B\sin\omega t\,\hat{j}$

Its a superposition of two SHM's, of unequal amplitude and phase difference of $\frac{\pi}{2}$. So, orbit is elliptical.

Velocity of particle, $\vec{v}=-A\omega\sin\omega t\,\hat{i}+B\omega\cos\omega t\,\hat{j}$

Speed $|\vec{v}|=\omega\sqrt{A^2\sin^2\omega t+B^2\cos^2\omega t}\neq$ constant

Angular momentum, $\vec{L}=m\vec{r}\times\vec{v}=m\omega AB\cos^2\omega t\,\hat{k}+AB\sin^2\omega t\hat{k}=mAB\omega\hat{k}$

Correct option is (a, c, d)

PRACTICE SET

1. If the projection of the vector $\vec{A}=\hat{i}+m\hat{j}+\hat{k}$ on vector $\vec{B}=4\hat{i}-4\hat{j}+7\hat{k}$ is equal to 19/9, then the value of 'm' is equal to

(a) 1 (b) -1 (c) 2 (d) -2

2. For any vector $\vec{A}$, which of the following statements is true?

(a) $\vec{A}=(\vec{A}\times\hat{i})\hat{i}+(\vec{A}\times\hat{j})\hat{j}+(\vec{A}\times\hat{k})\hat{k}$ (b) $\vec{A}=(\vec{A}\times\hat{i})\hat{i}+(\vec{A}\times\hat{j}).\hat{j}+(\vec{A}\times\hat{k})\hat{k}$

(c) $\vec{A}=(\vec{A}.\hat{i})\hat{i}+(\vec{A}.\hat{j})\hat{j}+(\vec{A}.\hat{k})\hat{k}$ (d) $\vec{A}=(\vec{A}.\hat{i})\times\hat{i}+(\vec{A}.\hat{j})\times\hat{j}+(\vec{A}.\hat{k})\times\hat{k}$

4. The value of the constant 'm' for which $\vec{A}=m\hat{i}+\hat{j}+\sqrt{5}\hat{k}$ subtends an angle 60^0 with $\vec{B}=4\hat{i}-5\hat{j}+\sqrt{5}\hat{k}$ is equal to

(a) $\sqrt{46/3}$ (b) $\sqrt{29/4}$ (c) $\sqrt{41/3}$ (d) $\sqrt{17/3}$

5. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\cos^{-1}(15/22)$ (b) $\cos^{-1}(225/2)$ (c) $\pi/2$ (d) $\pi/3$

6. If $\vec{a}$ and $\vec{b}$ are unit vectors, which of the following statements is true?

(a) $\sin\theta/2=\frac{1}{2}|\vec{a}-\vec{b}|$ (b) $\sin\theta/2=\frac{1}{2}|\vec{a}+\vec{b}|$ (c) $\cos\theta/2=\frac{1}{2}|\vec{a}+\vec{b}|$ (d) $\cos\theta/2=\frac{1}{2}|\vec{a}-\vec{b}|$

7. Given : $\vec{P}=2\hat{i}-3\hat{j}, \vec{Q}=\hat{i}+\hat{j}-\hat{k}, \vec{R}=3\hat{i}-\hat{k}$, the vector $\vec{V}$, which is orthogonal to $\vec{P}$ and $\vec{Q}$ and having unit scalar product with $\vec{R}$, can be estimated to be

(a) $(3\hat{i}+2\hat{j}+5\hat{k})/4$ (b) $(3\hat{i}+2\hat{j}-5\hat{k})/4$ (c) $(3\hat{i}-2\hat{j}+5\hat{k})/4$ (d) $(3\hat{i}-2\hat{j}-5\hat{k})/4$

8. The equation of the plane that contains the points P (2, -1, 1), Q(3, 2, –1), R(–1, 3, 2) is equal to

(a) $11x+5y+13z=30$ (b) $11x+7y+9z=30$

(c) $4x+7y-9z=30$ (d) $4x+11y-3z=30$

9. The unit vector which is perpendicular to vector $2\hat{i}+\hat{j}-3\hat{k}$ and lie in the plane of vectors $7\hat{i}-\hat{j}-\hat{k}$ and $\hat{i}+5\hat{j}-3\hat{k}$ is equal to

(a) $(3\hat{i}-3\hat{j}-\hat{k})/\sqrt{19}$ (b) $(3\hat{i}-3\hat{j}+\hat{k})/\sqrt{19}$ (c) $(3\hat{i}-3\hat{j}+\hat{k})$ (d) $(3\hat{i}-3\hat{j}-\hat{k})$

10. Which of the following relations is true?

(a) $(\vec{A}\times\vec{B}).(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})=0$ (b) $(\vec{A}\times\vec{B}).(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})=(\vec{A}.\vec{B}\times\vec{C})^2$

(c) $(\vec{A}\times\vec{B}).(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})=(\vec{A}\times(\vec{B}\times\vec{C}))^2$ (d) $(\vec{A}\times\vec{B}).(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})=(\vec{A}\times(\vec{B}\times\vec{C}))$

11. A particle moves along the curve $x=2t^2, y=t^2-4t$, $z=3t-5$. The components of its acceleration at t = 1 in the direction $\hat{i}-3\hat{j}+2\hat{k}$, is equal to

(a) $2/\sqrt{14}$ (b) $-2/\sqrt{14}$ (c) $16/\sqrt{14}$ (d) $-16/\sqrt{14}$

12. The unit tangent vector on the curve $x=t^2+1, y=4t-3, z=2t^2-6t$ at the point where $t=2$ is equal to

(a) $\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}$ (b) $\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}$ (c) $\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}$ (d) $\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}$

13. The unit vector perpendicular to the surface $x^2+y^2-z^2=11$ at the point (4, 2, 3) is equal to

(a) $4\hat{i}+2\hat{j}+3\hat{k}/\sqrt{29}$ (b) $4\hat{i}+2\hat{j}-3\hat{k}/\sqrt{29}$

(c) $4\hat{i}+2\hat{j}-3\hat{k}$ (d) $4\hat{i}+2\hat{j}+3\hat{k}$

14. The angle between the two surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at the point $(2,-1,2)$ is

(a) $\sin^{-1}\left(\frac{8}{3\sqrt{21}}\right)$ (b) $\cos^{-1}\left(\frac{8}{\sqrt{21}}\right)$ (c) $\sin^{-1}\left(\frac{8}{\sqrt{21}}\right)$ (d) $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

15. The directional derivative of $1/r$ in the direction of $\vec{r}$ is equal to

(a) $-1/r$ (b) $-1/r^2$ (c) $-1/r^3$ (d) $-1/r^4$

16. The directions along which there is no change in the value of the function $f(x,y)=e^{(x^2+xy)}$ at the point (3,-2) are equal to

(a) $0.6\hat{i}-0.8\hat{j}, -0.6\hat{i}+0.8\hat{j}$ (b) $-0.6\hat{i}-0.8\hat{j}, 0.6\hat{i}+0.8\hat{j}$

(c) $-0.6\hat{i}-0.8\hat{j}, 0.6\hat{i}-0.8\hat{j}$ (d) $0.6\hat{i}+0.8\hat{j}, -0.6\hat{i}+0.8\hat{j}$

17. The directional derivative of $\Omega(x,y)=\frac{x}{x^2+y^2}$ along a line making an angle 30^0 with the positive x-axis at (0,2)

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{\sqrt{3}}{8}$ (d) $\frac{1}{2\sqrt{2}}$

18. For a constant vector $\vec{a}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, consider the following statements:

$I.\ curl(\vec{a}\times\vec{r})=2[grad(\vec{a}.\vec{r})]$, $II.\ div[(\vec{a}\vec{r})\vec{r}]=2(\vec{a}\vec{r})$, then

(a) both statements are true (b) Only statement I is true

(c) Only statement II is true (d) Both statements are false

19. The value of the line integral $\int grad(x+y+z)\overline{dr}$ from (0,1,-1) to (1,2,0) is

(a) –1 (b) 3 (c) 0 (d) None of these

20. The work done by a force in moving a particle of mass 'm' form any point (x,y) to a neighbouring point (x+dx, y+dy) is given by $dW=2xy\,dx+x^2\,dy$. The work done for a complete cycle around a unit circle is

(a) 0 (b) 1 (c) 3 (d) 2π

21. If $\vec{F}=(5xy-6x^2)\hat{i}+(2y-4x)\hat{j}$, then the value of the line integral $\int_C \vec{F}.d\vec{r}$ along the curve C in the x-y plane given by $y=x^3$, from the point (1,1) to (2,8) is equal to

(a) 35 (b) -35 (c) 47 (d) -47

22. When a force $\vec{F}=(2x+y^2)\hat{i}+(3y-4x)\hat{j}$ is applied on a particle, then it starts to move along a right angled triangle PQR having vertices P(0,0), Q(2,0), R(2,1) respectively. The amount of work done by the force is equal to

(a) 14/3 (b) -14/3 (c) 22/3 (d) -22/3

23. If $\vec{A}=4xz\hat{i}-y^2\hat{j}+yz\hat{k}$, then the value of the integral $\int_S \vec{A}.\hat{n}dS$ (where S is the surface of a unit cube with two opposite corners at (0,0,0) and (1,1,1) respectively) is equal to

(a) 3/2 (b) 5/2 (c) 6 (d) 9/2

24. The value of the integral $\int_{(0,0)}^{(1,1)}\left[(3x^2+4xy+y^2)dx+(2x^2+2xy)dy\right]$ along the path $x^4+y^4=2xy$ is

(a) 0 (b) 1 (c) 2 (d) 4

25. Let 'S' be the Boundary of the region consiting of the parabolic cyllinder $z=1-x^2$ and the planes y = 0, y = 2, z = 0. The flux of the vector field $\vec{F}=xy\hat{i}+(y^2+e^{xz^2})\hat{j}+\sin(xy)\hat{k}$ through the surface 'S' is

(a) 2 (b) 4 (c) 8 (d) 16

26. Let $\vec{a}$ and $\vec{b}$ be two distinct three-dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by

(a) $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$ (b) $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{b^2}$ (c) $\frac{(\vec{a}.\vec{b})\vec{b}}{b^2}$ (d) $\frac{(\vec{b}.\vec{a})\vec{a}}{a^2}$

27. The equation of the plane that is tangent to the surface $xyz = 8$ at the point $(1, 2, 4)$ is

(a) $x + 2y + 4z = 12$ (b) $4x + 2y + z = 12$
(c) $x + 4y + 2z = 12$ (d) $x + y + z = 7$

28. A vector perpendicular to any vector that lies on the plane defined by $x + y + z = 5$, is

(a) $\vec{i} + \vec{j}$ (b) $\vec{j} + \vec{k}$ (c) $\vec{i} + \vec{j} + \vec{k}$ (d) $2\vec{i} + 3\vec{j} + 5\vec{k}$

29. If $\vec{A} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$, then the integral $\oint_C \vec{A}.d\vec{l}$ (where C is along the perimeter of a rectangular area bounded by $x = 0$, $x = a$ and $y = 0$, $y = b$) is

(a) $\frac{1}{2}\left(a^3 + b^3\right)$ (b) $\pi\left(ab^2 + a^2b\right)$ (c) $\pi\left(a^3 + b^3\right)$ (d) 0

ANSWER KEY

1. (d)	**2. (c)**	**3. (b)**	**4. (a)**	**5. (d)**	**6. (a, c)**	**7. (a)**
8. (a)	**9. (b)**	**10. (b)**	**11. (b)**	**12. (c)**	**13. (b)**	**14. (d)**
15. (b)	**16. (a)**	**17. (c)**	**18. (b)**	**19. (b)**	**20. (a)**	**21. (a)**
22. (b)	**23. (a)**	**24. (a)**	**25. (c)**	**26. (a)**	**27. (b)**	**28. (c)**
29. (d)						

30. Find the unit vector normal to the plane of two vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$; and sin of the angle between the vectors. [**Ans.** $\left(-3\hat{i} + 5\hat{j} + 11\hat{k}\right)/\sqrt{155}; \sqrt{155/156}$]

31. If the points P, Q, R are not lying on the same straight line having position vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ relative to a given origin, show that $\vec{x}\times\vec{y} + \vec{y}\times\vec{z} + \vec{z}\times\vec{x}$ is perpendicular to the plane containing P, Q, R.

32. A particle moves so that its position vector is given by, $\vec{r} = \cos\omega t\;\hat{i} + \sin\omega t\;\hat{j}$ where 'ω' is constant. Show that,
   (i) The velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.
   (ii) The acceleration is directed towards the origin and has magntidue proportional to the distance from the origin.
   (iii) $\vec{r} \times \vec{v}$ = constant vector.

33. If $\frac{d\vec{m}}{dt} = \vec{u} \times \vec{m}$ and $\frac{d\vec{n}}{dt} = \vec{u} \times \vec{n}$, then show that $\frac{d}{dt}(\vec{m}\times\vec{n}) = \vec{u}\times(\vec{m}\times\vec{n})$

34. If $f = x + y + z, g = x^2 + y^2 + z^2, h = xy + yz + zx$, the n show that $\left[\nabla f \nabla g \nabla h\right] = 0$.

35. Find the direction derivative of the scalar function $\phi = 4\,e^{(2x-y+z)}$ at the point $(1, +1, -1)$ in a direction towards the point $(-3, 5, 6)$. What will be its maximum value? [**Ans.** $-20/9$, $4\sqrt{6}$]

36. Show that $r^n\vec{r}$ is irrotational vector for any value of n, but it is solenoidal for $n = -3$ only.

37. Find an equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$. [**Ans.** $7x - 3y + 8z = 26$]

38. Given: $\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k}$ when $t = 2$ and $\vec{r}(t) = 4\hat{i} - 2\hat{j} + 3\hat{k}$ when t = 3. Show that, $\int_2^3 \vec{r}.\frac{d\vec{r}}{dt}\,dt = 10$

39. Evaluate $\int \vec{F}.d\vec{r}$ where $\vec{F} = C\left[-3a\sin\theta\cos\theta\hat{i} + a\left(2\sin\theta - 3\sin^2\theta\right)\hat{j} + b\sin 2\theta\hat{k}\right]$ and the curve is given by, $\vec{r} = a\cos\theta\hat{i} + a\sin\theta\,\hat{j} + b\theta\hat{k}$ (θ is varying from $\frac{\pi}{4}$ to $\frac{\pi}{2}$) [**Ans.** $\frac{1}{2}c\left(a^2 + b^2\right)$]

40. The acceleration of a particle at any time $t \geq 0$ is given by, $\vec{a} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$ If velocity $\vec{v}$ and displacement $\vec{r}$ are zero, at $t = 0$; find $\vec{v}$ and $\vec{r}$ at any time.

[**Ans.** (i) $6\sin 2t\,\hat{i} + (4\cos 2t - 4)\,\hat{j} + 8t^2\hat{k}$, (ii) $(3 - 3\cos 2t)\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3\hat{k}$]

41. If $\vec{A} = \left(3x^2 + 6y\right)\hat{i} - 14yz\,\hat{j} + 20\,xz^2\,\hat{k}$, evaluate $\int \vec{A}.d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the following paths:
   (i) $x = t, y = t^2, z = t^3$.
   (ii) The straight lines from $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)$
   (iii) The straight line joining (0, 0, 0) and (1, 1, 1) [**Ans.** (i) 5, (ii) 23/3, (iii) 13/3]

42. Show that, $\int_C \left[-\frac{y}{x^2 + y^2}\hat{i} + \frac{x}{x^2 + y^2}\hat{j}\right].d\vec{r} = 2\pi$, where '$C$' is the circle $x^2 + y^2 = 1$ in the x-y plane, described in counter clockwise direction.

43. Find $\int \vec{F}.d\vec{r}$ along the curve defined by $x^2 = 4y$ and $3x^2 = 8z$ from $x = 0$ to $x = 2$, where $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ [**Ans.** 441/40]

44. (i) Show that, $\vec{F} = \left(2xy + z^3\right)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force fields.
   (ii) Find the scalar potential.
   (iii) Find the work done in moving an object in this field from (1, –2, 1) to (3, 1, 4) [**Ans.** (ii) $x^2y + xz^3 + C$, (iii) 202 units]

45. Evaluate $\iint_S \vec{A}.\hat{n}ds$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.

46. Find the circulation of the field $\vec{F} = -x^2y\hat{i} + xy^2\hat{j} + \left(y^3 - x^3\right)\hat{k}$, around the curve the curve C, where C is the intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the plane $z = 3$. [**Ans.** 128π]

47. Using Gauss divergence theorem, evaluate $\iint_S \left(x^3 dydz + y^3 dzdx + z^3 dxdy\right)$, where 'S' is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ [**Ans.** $\frac{12}{5}\pi a^5$]

48. Consider the following vector field: $\vec{V} = xz^2\hat{i} - yz^2\hat{j} + z\left(x^2 - y^2\right)\hat{k}$

(a) Calculate the volume integral of the divergence of $\vec{V}$ over the region defined by $-a \le x \le a$, $-b \le y \le b$ and $0 \le z \le c$.

(b) Calculate the flux of $\vec{V}$ out of the region through the surface at $z = c$.

(c) Hence deduce the net flux through the rest of the boundary of the region.

[Ans. (a) $\frac{4}{3}abc\left(a^2 - b^2\right)$, (b) $\frac{4}{3}abc\left(a^2 - b^2\right)$, (c) 0]

49. Calculate the flux of the vector field $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ through the surface bounded by the region $x^2 + y^2 = 4, z = 0, z = 3$. **[Ans.** 84π]

50. Verify divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$, taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le 2a$, $0 \le z \le 3a/2$

51. Evaluate $\iint_s \left[y^2z^2 dydz + z^2x^2 dxdz + y^2z^2 dxdy\right]$ where S is the part of the spare $x^2 + y^2 + z^2 = 1$ above the $x - y$ plane.

52. Prove that $\iint_s \left(x^2\hat{i} + y^2\hat{j} + z^2\hat{k}\right)\cdot\overline{ds}$ vanishes where S denotes the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

53. Verify the Stokes' theorem for $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where is upper half surface of thes phere $x^2 + y^2 + z^2 = 1$ and 'C' is its boundary.

54. Evaluate $\iint_S \left(\nabla \times \vec{F}\right).\overline{ds}$, where $\vec{F} = (3x - 2y)\hat{i} + x^2z\hat{j} + y^2(z + 1)\hat{k}$ for a plane rectangular area with vertices at (0, 0), (1, 0), (1, 2), (0, 2) in the xy plane. **[Ans.** 4]

55. (a) Consider a constant vector field $\vec{v} = v_0\hat{k}$. Find any one of the many possible vectors $\vec{u}$ for which $\vec{\nabla} \times \vec{u} = \vec{v}$.

(b) Calculate the flux associated with the field $\vec{v}$ through the hemispherical surface defined by $x^2 + y^2 + z^2 = r^2, z > 0$

[Ans. (a) $v_0x\hat{j}$, (b) $\pi r^2 v_0$]

56. Evaluate $\int_C \left(x^2ydx + x^2dy\right)$, where 'C' is boundary described counter-clockwise direction of the triangle with vertices (0, 0), (1, 0), (1, 1). **[Ans.** 5/12]

57. Evaluate $\oint_c (e^x dx + 2ydy - dz)$ along $c: x^2 + y^2 = 4, z = 2$.

58. Evaluate $\int_c [yzdx + (xz + 1)dy + xydz]$, where c is any path from (1, 0, 0) → (2, 1, 4).

Chapter 2

MATRICES

2.1 Basic Review of Matrices

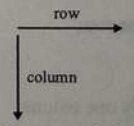
Definition:

A square or rectangular array of numbers or functions is known as matrices. Individual numbers or functions are known as elements of the matrix.

Example: $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 1-i & 2+i \\ 4+2i & x+iy \\ 2+4i & 3-2i \end{bmatrix}, \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

Order of a matrix:

A matrix of order $m \times n$ has 'm' number of rows and 'n' number of columns.



Every element of matrix is characterised by a row index and a column index:

a_{ij} : 'i' → row index, 'j' → column index, so, a_{ij} is the element of i^{th} row and j^{th} column

If a matrix A contains m rows and n columns, then it is written as $A = \left[a_{ij}\right]_{m\times n}$

Basic Review of matrix operation:

(i) Two matrices A and B are said to be equal if they are of the same order and each element of one is equal to corresponding element of the other matrix i.e. $a_{ij} = b_{ij}$ for all values of 'i' and 'j'.

(ii) Two matrices A and B are said to be conformable for addition/substraction if they have same order.

(iii) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\lambda A = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

(iv) Two matrices A and B are said to be comformable for multiplication if number of columns in A is equal to number of rows in B. Say, A matrix is of order '$m \times n$' and B matrix is of order '$p\times q$', then AB is possible only if $n = p$.

Various Type of Matrices:

(1) Square matrix: Matrix having number of rows equal to the number of columns i.e. $m = n$

Example: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

(2) Diagonal matrix: Matrix having elements other than the principal diagonal elements are zero

i.e $a_{ij} = 0$ for $i \neq j$.

Example: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

(3) Scalar matrix: Diagonal matrix having all principal diagonal elements to be same

i.e. $a_{ij} = 0$ for $i \neq j$ and a_{ii} is same for all i.

Example: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

(4) Identity/unit matrix: Diagonal matrix having all principal diagonal elements equal to one.

i.e. $a_{ij} = 0$ for $i \neq j$ and a_{ii} is equal to 1 for all i.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(5) Row Matrix: Matrix having only one row.

Example: $\begin{bmatrix} a & b & c \end{bmatrix}_{1\times 3}$

(6) Column matrix: Matrix having only one column.

Example: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1}$

(7) Null matrix: Matrix having all elements equal to zero i.e. $a_{ij} = 0$ for i, j

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(8) Upper triangular matrix: Matrix having elements below the principal diagonal equal to zero

i.e. $a_{ij} = 0$ for all $i > j$

Example: $\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$

(9) Lower triangular matrix: Matrix having elements above the principal diagonal equal to zero

i.e. $a_{ij} = 0$ for all $i < j$

Example: $\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix}$

(10) Periodic matrix: A square matrix 'A' for which $A^{k+1} = A$, is called periodic matrix of period k.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^5 = A$ i.e A is periodic matrix of period 4.

(11) Idempotent matrix: A square matrix 'A' for which $A^2 = A$, is called idempotent matrix. It is a periodic matrix of period 1. Any idempotent matrix will be either singular matrix or non-singular unit matrix.

Example: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(12) Nilpotent matrix: A square matrix for which $A^p = 0$, is called nilpotent matrix of index 'p'. The trace and determinant of the nilpotent matrix is always zero.

Example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix of index '2'.

(13) Involutory Matrix: A square matrix for which $A^2 = I$, is called involutory matrix. It is a self-inverse matrix.

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Trace of a matrix

It is defined as the sum of the principal diagonal elements of the matrix i.e.

$$Tr(A) = a_{11} + a_{22} + a_{33} + \ldots\ldots\ldots\ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Properties:

(i) $Tr(A+B) = Tr(A) + Tr(B)$　(ii) $Tr(cA) = cTr(A)$ [where c is constant]

(iii) $Tr(AB) = Tr(BA)$　(iv) $Tr(ABCD) = Tr(BCDA) = Tr(CDAB) = Tr(DABC)$

Transpose of a matrix:

By interchanging the rows and columns of a matrix, we will get the transpose of the original matrix.

Example: $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

Properties: (i) $(A^T)^T = A$, (ii) $(AB)^T = B^T A^T$

Transpose Conjugate of a Matrix:

It is obtained by taking the transpose of the matrix and then taking the complex conjugate of each element or vice versa i.e

$$A^{\dagger} = \left(A^{T}\right)^{*} = \left(A^{*}\right)^{T}$$

Example: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^{\dagger} = \begin{bmatrix} a^{*} & d^{*} & g^{*} \\ b^{*} & e^{*} & h^{*} \\ c^{*} & f^{*} & i^{*} \end{bmatrix}$

Properties: (i) $\left(A^{\dagger}\right)^{\dagger} = A$, (ii) $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$

Example 1: If each element of '$k \times k$' square matrix A can be chosen as a Boolean Variable, then one can construct 'M' number of different combinations of matrix 'A'. Therefore, 'M' is equal to

(a) $2k$ (b) 2^{k^2} (c) k^2 (d) 2^k

Soln: Total number of elements $= k^2$

Each of the elements can be chosen to be 0 or 1 i.e. in 2 ways

Therefore, entire k^2 number of elements can be chosen in $2 \times 2 \times \text{..........} k^2$ times $= 2^{k^2}$

Correct option is (b)

Example 2: Which of the following statements is NOT CORRECT?

(a) Trace of a unit matrix is always a fixed natural number
(b) Trace of the sum of two matrices is equal to sum of their traces
(c) Trace of the scalar matrix is some multiple of its order
(d) $Tr(AB) = Tr(BA)$

Soln: Trace of a unit matrix of order 'n' is equal to 'n' i.e. it is natural number but not fixed.

Correct option is (a)

Example 3: Consider the matrix $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then B^{1023} equals

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Soln: $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ and $B^4 = \left(B^2\right)^2 = (-I)^2 = I$

Therefore, $B^{1023} = B^{1020}.B^3 = \left(B^4\right)^{255}.B^3 = I.B^3 = B^3 = B^2.B = (-I).B = -B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Correct option is (b)

Symmetric matrix: Matrix for which $A^T = A$ i.e. $a_{ij} = a_{ji}$.

Example: $\begin{bmatrix} a & b & c \\ b & c & d \\ c & d & e \end{bmatrix}$

Note: The number of independent components of a $n \times n$ symmetric matrix is $\frac{1}{2}n(n+1)$.

Skew-symmetric matrix: Matrix for which $A^T = -A$ i.e. $a_{ij} = -a_{ji}$.

Example: $\begin{bmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{bmatrix}$

Note: (i) The number of independent components of a $n \times n$ antisymmetric matrix is $\frac{1}{2}n(n-1)$.

(ii) The principal diagonal elements of $n \times n$ skew-symmetric matrix are always zero.

- Every square matrix can be written as the sum of symmetric and antisymmetric matrix.

$$\mathrm{A} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

- If A and B are symmetric matrices, then AB will be symmetric if and only if A and B commute.

Hermitan matrix: Matrix for which $A^{\dagger} = A$ i.e. $a_{ij} = a_{ji}^{*}$.

Example: $A = \begin{bmatrix} a & d & e \\ d^{*} & b & f \\ e^{*} & f^{*} & c \end{bmatrix}$ where $a, b, c \in R$

Note: (i) Principal diagonal elements of Hermitian matrix are always real.

(ii) Minimum number of real elements for the most general $n \times n$ hermitian matrix is n

Skew-Hermitan matrix: Matrix for which $A^{\dagger} = -A$ i.e. $a_{ij} = -a_{ji}^{*}$.

Example: $A = \begin{bmatrix} a & d & e \\ -d^{*} & b & f \\ -e^{*} & -f^{*} & c \end{bmatrix}$ where a, b, c can be either zero or purely imaginary.

Note: (i) Principal diagonal elements of a skew-hermitian matrix are either zero or purely imaginary.

(ii) Minimum number of real elements in a most general $n \times n$ skew hermitian matrix is zero.

(iii) If A is Hermitian matrix, iA and $-iA$ will be anti-Hermitian.

(iv) If A is Hermitian matrix, kA (k is real scalar) will be also Hermitian.

Example 4: The product MN of two Hermitian matrices M and N is anti-Hermitian. It follows that

(a) $\{M,N\}=0$ (b) $[M,N]=0$ (c) $M^{\dagger}=N$ (d) $M^{\dagger}=N^{-1}$

Soln: If MN is anti-Hermitian, then

$$(MN)^{\dagger}=-(MN)\Rightarrow N^{\dagger}M^{\dagger}=-MN\Rightarrow NM=-MN\Rightarrow MN+NM=0\Rightarrow\{M,N\}=0$$

Correct option is (a)

Singular matrix: Matrix having zero determinant.

Example: $\begin{bmatrix}2&1&5\\3&5&7\\2&1&5\end{bmatrix}$

Non-singular matrix: Matrix having non-zero determinant.

Example: $\begin{bmatrix}3&1&1\\6&-2&4\\0&3&-7\end{bmatrix}$

Adjoint of a Matrix: Adjoint of a matrix is defined as the transpose of co-factor matrix.

If $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$, then $Adj\ A=\begin{bmatrix}c_{11}&c_{21}&c_{31}\\c_{12}&c_{22}&c_{32}\\c_{13}&c_{23}&c_{33}\end{bmatrix}$

Note: (i) $(Adj\ A).\ A= A.(Adj\ A) = |A|\ I_n$

(ii) If A is a square matrix of order 'n' having determinant 'm', then $|\ Adj\ A\ | = m^{n-1}$.

Inverse of a Matrix:

Let, A be a square matrix of order 'n', and B be a square matrix of order 'n' such that $AB = BA = I_n$,

then $\boxed{B = A^{-1} = \text{inverse of matrix } A = \frac{adj\ A}{|A|}}$; Therefore, A is called invertible matrix.

Note: (i) Any invertible matrix possesses an unique inverse.

(ii) The necessary and sufficient condition for a square matrix A to posses an inverse is that it is non-singular.

(iii) $(ABC....)^{-1} = ...\ C^{-1}B^{-1}A^{-1}$

Example 5: If 'I' is an identity matrix of order '$n \times n$' and adj $(2I) = 2^k\,\mathrm{I}$, then 'k' is equal to

(a) 1 (b) 2 (c) n-1 (d) n

Soln. $A.(adj\ A)=|A|I\Rightarrow(2I).(adj\ 2I)=|2I|I\Rightarrow 2(adj\ 2I)=2^nI\Rightarrow(adj\ 2I)=2^{n-1}I\Rightarrow k=n-1$

Correct option is (c)

Example 6: Find the adjoint of $A=\begin{bmatrix}1&2&3\\2&-3&1\\3&1&-2\end{bmatrix}_{3\times3}$

$Adj\ A=\begin{bmatrix}c_{11}&c_{21}&c_{31}\\c_{12}&c_{22}&c_{32}\\c_{13}&c_{23}&c_{33}\end{bmatrix}$; Now, $c_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix}-3&1\\1&-2\end{vmatrix}=5$,

Similarly $c_{12}=7, c_{13}=11, c_{21}=7, c_{22}=-11, c_{23}=5, c_{31}=11, c_{32}=5, c_{33}=-7$

$\therefore\ Adj\ A=\begin{bmatrix}5&7&11\\7&-11&5\\11&5&-7\end{bmatrix}$

Example 7: If 'A' is a 3×3 matrix with determinant 3, then the determinant of $adj.[adj.[adj.(A^{-1})]]$ is equal to

(a) $\frac{1}{81}$ (b) $\frac{1}{729}$ (c) $\frac{1}{6561}$ (d) $\frac{1}{243}$

$$|A|=3\Rightarrow|A^{-1}|=\frac{1}{3}\Rightarrow\left|adj\left(A^{-1}\right)\right|=\left(\frac{1}{3}\right)^{3-1}=\frac{1}{9}\Rightarrow\left|adj\left\{adj\left(A^{-1}\right)\right\}\right|=\left(\frac{1}{9}\right)^{3-1}=\frac{1}{81}$$

$$\Rightarrow\left|adj\left[adj\left\{adj\left(A^{-1}\right)\right\}\right]\right|=\left(\frac{1}{81}\right)^{3-1}=\frac{1}{6561}$$

Correct option is (c)

Orthogonal Matrix: A square matrix 'A' is said to be orthogonal if $AA^T = A^TA = I$

Note: (i) Determinant of an orthogonal matrix will be either +1 or –1.

(ii) $A^T = A^{-1}$.

(iii) Each row and column of an orthogonal matrix is a normalized vector.

(iv) Any two rows or any two columns of an orthogonal matrix are orthogonal to each other.

(v) If A and B are orthogonal matrices, then AB and BA will be always orthogonal.

Example: $A=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$

Unitary matrix: A square matrix 'A' is said to be orthogonal if $AA^{\dagger}=A^{\dagger}A=I$

Note: (i) Determinant of an orthogonal matrix will be of unit modulus..

(ii) $A^{\dagger}=A^{-1}$

(iii) Each row and column of an unitary matrix is a normalized vector.

(iv) Any two rows or any two columns of an unitary matrix are orthogonal to each other.

(v) If A and B are unitary matrices, then AB and BA will be always unitary.

Example: $A=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 8: The matrix A, defined by $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix}$ is orthogonal if

(a) $a=1, b=-1$ (b) $a=1/\sqrt{2}, b=-1/\sqrt{2}$ (c) $a=1/\sqrt{2}, b=-i/\sqrt{2}$ (d) $a=1,\ b=1$

Soln: Applying the condition for the orthogonal matrices i.e. $AA^T=I$ we get, $a^2+b^2=1$

Only option (b) satisfies this.

Correct option is (b)

2.2 Eigenvalues and Eigenvectors:

Consider the following linear transformation,

$$AX=\lambda X \Rightarrow (A-\lambda I)\,X=0$$

The values of λ for which the above equation has non zero solution of X $(X \neq 0)$ is called eigen valu or characteristic root of matrix A and corresponding non-zero solution X is called eigenvector o characteristic vector of matrix A corresponding to that value of λ.

Here, $(A-\lambda I)$ is called Characteristic matrix of matrix A, $|A-\lambda I|$ is called Characteristic polynomi of matrix A, and $|A-\lambda I|=0$ is called Characteristic equation or eigen equation of matrix A.

Example 9: Find the characteristic roots or eigenvalues and eigenvectors of matrix.

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Soln: Eigenvalue equation:

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}=0 \Rightarrow (1-\lambda)\left[(2-\lambda)^2-0\right]=0 \Rightarrow \lambda=1,2,2$$

Calculation of eigenvectors:

(i) $\lambda_1=1$: $AX_1=\lambda_1 X_1$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1+2x_2+3x_3 \\ 2x_2+3x_3 \\ 2x_3 \end{pmatrix}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{aligned} x_1+2x_2+3x_3&=x_1 \\ 2x_2+3x_3&=x_2 \\ 2x_3&=x_3 \end{aligned}$$

Solving these equations, we get

$x_3=0,\ x_2=0$ & let $x_1=k$. Therefore, $X_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$

If this eigen vector is normalized to unity then,

$$X_1^{\dagger}X_1=1 \Rightarrow \begin{bmatrix} k^* & 0 & 0 \end{bmatrix}\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}=1 \Rightarrow |k|^2=1 \Rightarrow k=1 \Rightarrow X_1=\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(ii) $\lambda_2=\lambda_3=2$:

Let the eigen vector is $X_2=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\therefore\ AX_2=\lambda_2 X_2$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1+2x_2+3x_3 \\ 2x_2+3x_3 \\ 2x_3 \end{pmatrix}=\begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} \Rightarrow \begin{aligned} x_1+2x_2+3x_3&=2x_1 \\ 2x_2+3x_3&=2x_2 \\ 2x_3&=2x_3 \end{aligned}$$

it gives $x_1=2x_2$ & x_3 can be arbitary, we choose x_3 to be zero.

Let $x_2=k \Rightarrow x_1=2k$ $\therefore$ $X_2=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 2k \\ k \\ 0 \end{pmatrix}$

If the eigen vector is normalized to unity, then

$$X_2^{+}X_2=1 \Rightarrow \begin{pmatrix} 2k^* & k^* & 0 \end{pmatrix}\begin{pmatrix} 2k \\ k \\ 0 \end{pmatrix}=1 \Rightarrow 5|k|^2=1 \Rightarrow k=\frac{1}{\sqrt{5}}, \text{ therefore, } X_2=\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}$$

Note: (i) Sum of the eigenvalues of a matrix is equal to trace of the Matrix i.e.

$$\lambda_1+\lambda_2+\lambda_3+\ldots\ldots\ldots\ldots+\lambda_n=Tr(A)$$

(ii) Product of the eigenvalues of a matrix is equal to determinant of the matrix i.e.

$$\lambda_1.\lambda_2.\lambda_3\ldots\ldots\ldots\ldots\lambda_n=Det(A)$$

(iii) Any square matrix A and its transpose have same eigen values.

(iv) If $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the eigenvalues of A, then

(a) eigenvalues of the matrix kA will be $k\lambda_1, k\lambda_2, k\lambda_3, \ldots$

(b) eigenvalues of the matrix A^{-1} will be $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \ldots$

(c) eigenvalues of the matrix A^m will be $\lambda_1^m, \lambda_2^m\ \lambda_3^m \ldots$

(v) Eigenvalues of a real symmetric matrix or a Hermitian matrix are always real.

(vi) Eigen values of a real skew-symmetric matrix or a skew-Hermitian matrix are either zero or purely imaginary.

(vii) Eigen values of an real orthogonal matrix or Unitary matrix are of unit modulus.

(viii) Eigen values of a diagonal matrix or a upper triangular matrix or a lower triangular matrix are the principal diagonal elements.

(ix) Any two eigen vectors corresponding to two distinct eigen values of Hermitian matrix and a Unitary matrix are orthogonal to each other.

(x) Eigenvalues of a matrix is invariant under similarity transformation.

(xi) Eigenvalues of a nilpotent matrix are alwayszero.

(xii) Eigenvalues of idempotent matrix are either zero or unity.

(xiii) Consider a $n \times n$ matrix having all elements equal to one. One of the eigenvalue of the matrix will be equal to order of the matrix and all other eigenvalues are zero i.e $n, 0, 0, 0, \ldots$..

(xiv) Consider a $n \times n$ matrix having rows and columns which are scalar multiple of a particular row and column respectively. One of the eigenvalue of the matrix will be equal to trace of the matrix and all other eigenvalues are zero i.e trace,$0, 0, 0, \ldots\ldots\ldots$

Example 10: If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigenvalues of $3A^3 + 5A^2 - 6A + 2I$.

ln: A is an upper triangular matrix, therefore eigenvalues of matrix A will be $\lambda = 1, 3, -2$

Eigen values of $3A^3 = 3, 81, -24$

Eigen values of $5A^2 = 5, 45, 20$

Eigen values of $6A = 6, 18, -12$

Eigen values of $2I = 2, 2, 2$

Eigenvalue of given expression will be

$$(3+5-6+2)=4, (81+45-18+2)=110, (-24+20+12+2)=10$$

Example 11: The eigenvalues of $\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$ are

(a) 1, 3 (b) –1, –6 (c) 1, –6 (d) –2, –5

ln: Sum of the eigenvalues is equal to the trace of matrix i.e. $\lambda_1 + \lambda_2 = -7$

Product of the eigenvalues is equal to the determinant of matrix i.e. $\lambda_1\lambda_2 = 6$

Only option (b) satisfies this.

Therefore, eigenvalues are $= -1, -6$

Correct option is (b)

Example 12: If trace of matrix M of order 3, is 13 and its determinant is 36, then the largest eigenva of M is equal to

(a) 4 (b) 6 (c) 9 (d) 12

Soln: Sum of the eigenvalues = trace of the matrix i.e. $\lambda_1 + \lambda_2 + \lambda_3 = 13$

Product of the eigenvalues = determinant of the matrix i.e. $\lambda_1\lambda_2\lambda_3 = 36$

Only the combiantion $\lambda_1 = 9, \lambda_2 = 2, \lambda_3 = 2$ satisfies the both conditions simultaneously.

Correct option is (c)

Example 13: What are the number of zero eigenvalues of matrix $\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_5^2 \end{pmatrix}$?

Soln: For the given matrix, Second row $= \frac{a_2}{a_1} \times$ first row & third row $= \frac{a_3}{a_1} \times$ first row

Therefore, the eigenvalues of the matrix = trace of the matrix, $0, 0 = \left(a_1^2 + a_2^2 + a_3^2\right), 0, 0$

2.3 Cayley-Hamilton Theorem:

Every square matrix satisfies its own characteristic equation or eigenvalue equation.

Example: $A = \begin{bmatrix} 1 & 2 \\ 4 & -6 \end{bmatrix} \Rightarrow$ It's characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & -6-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+6)(\lambda-1)-8=0 \Rightarrow \lambda^2+5\lambda-14=0$$

Therefore, Matrix A satisfies the above equation i.e. $A^2 + 5A - 14I = 0$

Calculation of the inverse of the given matrix:

Since, $A^2 + 5A = 14I \Rightarrow A^{-1}A^2 + 5A^{-1}A = 14A^{-1}I \Rightarrow A^{-1} = \frac{1}{14}(A+5I)$

Example 14: Let P be a n×n diagonalizable matrix which satisfies the equations

$$P^2 = P, Tr(P) = n-1$$

Det (P) is :

(a) n (b) 0 (c) 1 (d) $n-1$

Soln: Using Cayley-hamilton theorem, $P^2 - P = 0 \quad \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0 \qquad \Rightarrow \lambda = 0, 1$

$\det(M) = \lambda_1 . \lambda_2 \ldots\ldots\ldots\ldots \lambda_n = 0$ (Since, one of the eigenvalue is zero)

Example 15: Let M be a 3×3 Hermitian matrix which satisfies the matrix equation

$$M^2 - 5M + 6I = 0$$

Where I refers to the identity matrix. Which of the following are possible eigenvalues of M

(a) $\{1, 2, 3\}$ (b) $\{2, 2, 3\}$ (c) $\{2, 3, 5\}$ (d) $5, 5, 6\}$

Soln: According to Cayley-Hamilton theorem, we can write $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2,3$

Correct option is (b)

Example 16: The eigenvalues of matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ are

(a) $+1, +1, +1$ and $+1$ (b) $+1, +1, -1$ and -1

(c) $0, 0, \sqrt{2}$ and $-\sqrt{2}$ (d) $+\sqrt{2}, -\sqrt{2}, +1/\sqrt{2}$ and $-1/\sqrt{2}$

Soln: The given matrix A satisfies the relation $A^2 = I$

According to Cayley-Hamilton theorem, we can write $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

Correct option is (b)

Example 17: The trace of a real 4 x 4 matrix U = exp(A), where $A = \begin{pmatrix} 0 & 0 & 0 & \frac{\pi}{4} \\ 0 & 0 & -\frac{\pi}{4} & 0 \\ 0 & \frac{\pi}{4} & 0 & 0 \\ -\frac{\pi}{4} & 0 & 0 & 0 \end{pmatrix}$ is

(a) $2\sqrt{2}$ (b) $\frac{\pi}{4}$ (c) 0 (d) 2

Soln: $A^2 = -\frac{\pi^2}{16} I; A^3 = A^2 . A = -\frac{\pi^2}{16} A; A^4 = \left(A^2\right)^2 = \frac{\pi^4}{256} I$; and so on ....

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots\ldots\ldots\ldots = I + A - \frac{1}{2!}\frac{\pi^2}{16} I - \frac{1}{3!}\frac{\pi^2}{16} A + \frac{1}{4!}\frac{\pi^4}{256} I + \ldots\ldots$$

$$Tr\left[\exp(A)\right] = Tr(I) + Tr(A) - Tr\left(\frac{1}{2!}\frac{\pi^2}{16} I\right) - Tr\left(\frac{1}{3!}\frac{\pi^2}{16} A\right) + Tr\left(\frac{1}{4!}\frac{\pi^4}{256} I\right) + \ldots\ldots\ldots$$

$$= 4 + 0 - 4.\frac{1}{2!}\frac{\pi^2}{16} - 0 + 4.\frac{1}{4!}\frac{\pi^4}{256} + \ldots\ldots\ldots\ldots = 4\left[1 - \frac{1}{2!}\frac{\pi^2}{16} + \frac{1}{4!}\frac{\pi^4}{256} - \ldots\ldots\right] = 4\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}$$

2.4 Rank of a matrix

A natural number ‘r’ is said to be the rank of the matrix A if it has the following properties:

(i) There exists at least one squre sub-matrix of the matrix A of order ‘r’ having non-zero determinant.

(ii) Every square sub-matrix of matrix A of order ‘$r + 1$’ have zero determinant.

i.e. Rank of matrix ≡ order of the largest square sub-matrix of the given matrix having non-vanishing determinant.

Note: (i) The rank of the null matrix is zero.

(ii) The rank of any non-zero matrix is ≥ 1.

(iii) The rank of any non-singular matrix of order ‘n’ is ‘n’.

(iv) The rank of a product of two matrices cannot exceed the rank of either matrix.

Example 18: Find the rank of the matrix, (i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 4 \\ 5 & 1 & 2 \end{bmatrix}$, (ii) $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

Soln: (i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 4 \\ 5 & 1 & 2 \end{bmatrix} \Rightarrow |A| = 3(6-4) - 2(0-20) + (-1)(0-15) = 6 + 40 + 15 = 61 \neq 0$; So, $\rho(A) = 3$

(ii) $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix} \Rightarrow |B| = 2(-9+8) - 1(0+4) + (-1)(0-6) = -2 - 4 + 6 = 0$; So, $\rho(B) \neq 3$

Now, a 2 × 2 sub-matrix of B: $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, it’s determinant = 6 , So, $\rho(B) = 2$

Example 19: The rank of matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$ (a, b, c, being all real), is 3 if

(a) $a = b = c$

(b) a, b, c are all different but $a + b + c = 0$

(c) Two of the numbers a, b, c are equal but are different from the third

(d) a, b, c are all different and $a + b + c \neq 0$

Soln: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$ (applying $C'_3 = C_3 - C_1, C'_2 = C_2 - C_1$) $= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & b^2+ab+a^2 & c^2+ca+a^2 \end{vmatrix} = (b-a)(c-a)(b-c)(a+b+c)$$

The rank of the matrix is 3 if all a, b, c are different and $a + b + c \neq 0$

Correct option is (d)

2.5 Diagonalization of Matrix

Let, $\lambda_1, \lambda_2, \lambda_3 ....\lambda_n$ be 'n' distinct eigen values of a diagonalisable matrix A and X_1, $X_2 ... X$ corresponding linearly independent eigenvectors of the matrix A.

Diagonalized matrix D corresponding to A: $D = P^{-1} AP$

where matrix P is given by,

$$P = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ x_{31} & x_{32} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \text{ where } X_I = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$$

Example 20: Diagonalize the matrix $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Soln: Eigenvalue equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \quad \lambda = \pm 1 \text{ i.e. } \lambda_1 = 1,\ \lambda_2 = -1$$

Now the eigenvector corresponding to λ_1 is let X_1.

Therefore, $AX_1 = \lambda_1 X_1$

$$\Rightarrow \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} -ix_2 \\ ix_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x_1 = -ix_2 \ \&\ x_2 = ix_1$$

Let $x_1 = k,\ x_2 = ik \qquad \therefore\ X_1 = \begin{pmatrix} k \\ ik \end{pmatrix}$

Using the normalization condition i.e. $X_1^\dagger X_1 = 1$

$$\Rightarrow \quad \begin{pmatrix} k^* & -ik^* \end{pmatrix}\begin{pmatrix} k \\ ik \end{pmatrix} = 1 \quad \Rightarrow \quad |k|^2 + |k|^2 = 1 \quad \Rightarrow \quad k = \frac{1}{\sqrt{2}}$$

This gives $\quad X_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ +\frac{i}{\sqrt{2}} \end{pmatrix}$

Similarly, eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is $\quad X_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

Now, Diagonalizing matrix $P = (X_1 \ \ X_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$

This matrix P is the matrix which can diagonalize A and the diagonalized matrix will be

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Altrenative Method:

(i) First find the eigen value of A i.e. $\lambda_1, \lambda_2 ....$

(ii) Now, if all eigenvalues distinct (no two eigenvalues are equal), then putting eigenvalues in place of princ
diagonal elements and write zero for all off diagonal elements, we will get D as

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & \cdots & \cdots \\ 0 & 0 & \ddots & \cdots \\ \vdots & \vdots & \vdots & \lambda_n \end{bmatrix}$$

Calculation of Power of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \quad \Rightarrow A = PDP^{-1}$

$$\Rightarrow A^K = PD^KP^{-1} \text{ (} K \text{ is a positive integer)}$$

Calculation of Exponential of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \quad \Rightarrow A = PDP^{-1}$

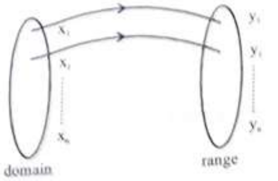
$$\Rightarrow (\exp A) = P\,(\exp D)\,P^{-1}$$

Calculation of Logarithm of a Matrix:

According to similarity tranformation, $D = P^{-1}AP \quad \Rightarrow A = PDP^{-1}$

$$\Rightarrow (\ln A) = P\,(\ln D)\,P^{-1}$$

2.6 Linear transformation:



There is one to one correspondence between the elements of domain and range. Thus $y = f(x)$

$f(x)$ is the transformation rule relating domain and range.

If any transformation follows the rules below :

(i) $f(x_1 + x_2) = f(x_1) + f(x_2)$

(ii) $f(cx_1) = cf(x_1)$

then it is called linear transformation.

Example : $y = f(x) = ax$

$$f(x_1 + x_2) = a(x_1 + x_2) + b = ax_1 + ax_2 = f(x_1) + f(x_2)$$

Therefore, it is a linear operator.

The techniqual definition of linear transformation is:

Linear transformation is a maping from one vector space to another which follows the property

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

where 'a', 'b' are constants and belong to a field.

Matrix representation of Linear transformation:

Let $2x_1 + 3x_2 - x_3 = 2;\ \ 5x_1 + 6x_2 - 3x_3 = 10;\ \ x_1 + x_2 + x_3 = 8$

L.H.S. of the equations can be represented by a linear transformation.

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The reqaution can be written as $TX = Y$ where, T is a $n \times n$ matrix and this shows the effect of T on th

column matrix $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Example 21: Show that $T(x, y) = (2x + 3y, 3x - 4y)$ is a linear transformation.

Soln: Let $T(a\alpha + b\beta) = T[a(x_1, y_1) + b(x_2, y_2)] = T(ax_1 + bx_2, ay_1 + by_2)$

$$= [2(ax_1 + bx_2) + 3(ay_1 + by_2), 3(ax_1 + bx_2) - 4(ay_1 + by_2)]$$

$$= [2ax_1 + 2bx_2 + 3ay_1 + 3by_2, 3ax_1 + 3bx_2 - 4ay_1 - 4by_2]$$

and $aT(\alpha) + bT(\beta) = aT(x_1, y_1) + bT(x_2, y_2)$

$$= a(2x_1 + 3y_1, 3x_1 - 4y_1) + b(2x_2 + 3y_2, 3x_2 - 4y_2)$$

$$= [2ax_1 + 2bx_2 + 3ay_1 + 3by_2, 3ax_1 + 3bx_2 - 4ay_1 - 4by_2]$$

$\therefore$ $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

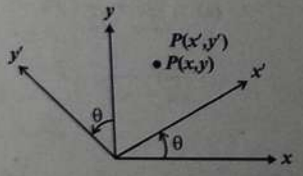
Therefore, the transformation is linear.

2.7 Applications

Rotation Matrix:

Let (x, y, z) be a cartesian coordinate system in a three dimension space. Let $\vec{r} = x\hat{i} + y\hat{j}$ be a vector in x-y plane. If we consider a rotation of coordinate system about z-axis through an angle θ in anticlockwise sense and indicate the new axis by (x', y'), then the components of the same vector $\vec{r} = x'\hat{i}' + y'\hat{j}'$ relative to the new system may be expressed as

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$



Then equations in matrix form can be expressed as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = R_z(\theta)\begin{bmatrix} x \\ y \end{bmatrix}$$

where $R_z(\theta)$ is the transformation matrix corresponding to the rotation in x-y plane.

Note: (i) $R_z(\theta)$ is orthogonal matrix.

(ii) Determinant of $R_z(\theta)$ is 1.

(iii) Eigenvalues of $R_z(\theta)$ are $e^{i\theta}$ and $e^{-i\theta}$.

Let us consider a vector in space i.e. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Here, three types of rotation will be possible:

CASE I: Rotation about z-axis:

Rotation matrix. $R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

CASE II: Rotation about x-axis:

$$\text{Rotation matrix. } R_x(\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}$$

CASE III: Rotation about y-axis:

$$\text{Rotation matrix. } R_y(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$$

Example 22: Consider a vector $\vec{P}=\hat{i}+2\hat{j}+\hat{k}$ in the co-ordinate system $(\hat{i},\hat{j},\hat{k})$. The axes are in the clockwise direction about the z axis by an angle 45^0. The vector $\vec{P}$ in the rotated co- system will be

(a) $-\frac{1}{\sqrt{2}}\hat{i}'+\frac{3}{\sqrt{2}}\hat{j}'+\hat{k}'$ (b) $\frac{1}{\sqrt{2}}\hat{i}'-\frac{3}{\sqrt{2}}\hat{j}'+\hat{k}'$

(c) $\frac{3}{\sqrt{2}}\hat{i}'+\frac{1}{\sqrt{2}}\hat{j}'+\hat{k}'$ (d) $\frac{3}{\sqrt{2}}\hat{i}'-\frac{1}{\sqrt{2}}\hat{j}'+\hat{k}'$

Soln: $$\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix}=\begin{bmatrix}\cos 45^0 & -\sin 45^0 & 0\\ \sin 45^0 & \cos 45^0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\cos 45^0 & -\sin 45^0 & 0\\ \sin 45^0 & \cos 45^0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{3}{\sqrt{2}}\\ 1\end{bmatrix}$$

Correct Option is (a)

Existance and uniqueness of the solution of system of Simultaneous Linea Homogeneous Equations:

Consistent Equations: They have one or more solutions.

Example: (a) Unique Solution: $x+2y=4; 3x+2y=2$

(b) Infinite Solution: $x+2y=4; 3x+6y=12$

Inconsistent Equations: They have no solution.

Example: $x+2y=4; 3x+6y=5$

Suppose a system of 'n' simultaneous linear equations containing 'n' unknown variables as following:

$$\begin{aligned}&a_{11}x_1+a_{12}x_2+\dots\dots+a_{1n}x_n=b_1\\ &a_{21}x_1+a_{22}x_2+\dots\dots+a_{2n}x_n=b_2\\ &\dots\dots\dots\dots\dots\dots\\ &\dots\dots\dots\dots\dots\dots\\ &a_{n1}x_1+a_{n2}x_2+\dots\dots+a_{nn}x_n=b_n\end{aligned}$$

$$\text{where co-efficient matrix of the system} = A=\begin{bmatrix}a_{11} & a_{12} & \dots & \dots & a_{n1}\\ a_{21} & a_{22} & \dots & \dots & a_{n2}\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ a_{n1} & a_{n2} & \dots & \dots & a_{nn}\end{bmatrix}$$

$$\text{and Augamented matrix of the system} = C=\begin{bmatrix}a_{11} & a_{12} & \dots & \dots & a_{n1} & b_1\\ a_{21} & a_{22} & \dots & \dots & a_{n2} & b_2\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} & b_n\end{bmatrix}$$

(1) If rank of A = rank of C, then the equations are consistent.
   (a) If rank of A = rank of C = number of unknown variables, then there will be unique solution.
   (b) If rank of A = rank of C < number of unknown variables, then there will be infinite solution.
(2) If rank of A $\neq$ rank of C, then the equations are inconsistent and have no solutions.

CSIR-NET/JRF Previous Years Question with Solutions

1. Consider the matrix $M=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ **[CSIR June - 2011]**

A. The eigenvalues of M are

(a) 0,1,2 (b) 0,0,3 (c) 1,1,1 (d) -1,1,3

Soln. Eigenvalues of a n x n matrix having all elements equal to one, are n,0,0,0,........... respectively.

Therefore, eigen values of M are : 0, 0, 3

Correct answer is (b)

B. The exponential of M simplifies (I is the 3 x 3 identity matrix)

(a) $e^M=I+\left(\frac{e^3-1}{3}\right)M$ (b) $e^M=I+M+\frac{M^2}{2!}$ (c) $e^M=I+3^3M$ (d) $e^M=(e-1)M$

Soln. $e^M=I+M+\frac{M^2}{2!}+\frac{M^3}{3!}+.........$

Now, $M^2=\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix}=3M, M^3=\begin{pmatrix}9&9&9\\9&9&9\\9&9&9\end{pmatrix}=3^2M$ and so on

Therefore, $e^M=I+M+\frac{3M}{2!}+\frac{3^2M}{3!}+.........=I+\frac{M}{3}\left(3+\frac{3^2}{2!}+\frac{3^3}{3!}+......\right)=I+\frac{M}{3}\left(e^3-1\right)$

Correct answer: (a)

2. A 3 x 3 matrix M has $Tr[M]=6, Tr[M^2]=26, Tr[M^3]=90$. Which of the following can be possible set of eigenvalues of M? **[CSIR Dec - 2011]**

(a) {1,1,4} (b) {-1,0,7} (c) {-1,3,4} (d) {2,2,2}

Soln. $\lambda_1+\lambda_2+\lambda_3=6;\quad \lambda_1^2+\lambda_2^2+\lambda_3^2=26;\ \lambda_1^3+\lambda_2^3+\lambda_3^3=90$

It is trivial to check that only $\{-1,3,4\}$ satisfies there three equation.

Correct answer is (c).

3. The eigenvalues of the matrix $A=\begin{pmatrix}1&2&3\\2&4&6\\3&6&9\end{pmatrix}$ **[CSIR June - 2012]**

(a) (1,4,9) (b) (0,7,7) (c) (0,1,13) (d) (0,0,14)

Soln. Eigenvalues of a n x n matrix having rows and columns as a scalar multiple of a particular row and column respectively, are trace,0,0,0,............

Here, the second row and third row are twice and thrice of the first row and same is true for the columns.

So, the eigenvalues of the given matrix are 14, 0, 0

Correct answer is (d)

4. The eiegenvalues of the anti-symmetric matrix $A=\begin{pmatrix}0&-n_3&n_2\\n_3&0&-n_1\\-n_2&n_1&0\end{pmatrix}$ where n_1,n_2,n_3 are components of an unit vector, are **[CSIR June - 2012]**

(a) 0, -i, i (b) 0, 1, -1 (c) 0, 1+i, -1-i (d) 0, 0, 0

Soln. Characteristic equation of the given matrix : $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}-\lambda&-n_3&n_2\\n_3&-\lambda&-n_1\\-n_2&n_1&-\lambda\end{vmatrix}=0\Rightarrow\lambda^3+\lambda\left(n_1^2+n_2^2+n_3^2\right)=0$$

Since, n_1,n_2,n_3 are the components of an unit vector, so $n_1^2+n_2^2+n_3^2=0$

$\Rightarrow\lambda^3+\lambda=0\quad\Rightarrow\lambda=0,i,-i$

Correct answer is (a)

5. A 2×2 matrix 'A' has eigenvalues $e^{i\pi/5}$ and $e^{i\pi/6}$. The smallest value of 'n' such that $A^n=I$ **[CSIR Dec - 2012]**

(a) 20 (b) 30 (c) 60 (d) 120

Soln. A has eigenvalues $e^{i\frac{\pi}{5}}$ and $e^{i\frac{\pi}{6}}$

A^n has eigenvalues $e^{i\frac{n\pi}{5}}$ and $e^{i\frac{n\pi}{6}}$

So, $|A^n|=$ product of eigenvalues $=e^{i\frac{11n\pi}{30}}$

Since, $A^n=I\quad\Rightarrow|A^n|=1$

$\therefore\quad e^{i(11n\pi/30)}=1=e^{2m\pi i}\Rightarrow\frac{11n\pi}{30}=2m\pi\Rightarrow\frac{n}{m}=\frac{60}{11}$

So, the smallest value of 'n' is 60.

Correct answer is (c)

Given a 2 x 2 unitary matrix satisfying $U^\dagger U = UU^\dagger = I$ with $\det U = e^{i\varphi}$, one can construct a unitary matrix V $(V^\dagger V = VV^\dagger = I)$ with det V = 1 from it by [CSIR Dec - 2012]

(a) Multiplying U by $e^{-i\varphi/2}$
(b) Multiplying a single element of U by $e^{-i\varphi}$
(c) Multiplying any row or column by $e^{-i\varphi/2}$
(d) Multiplying U by $e^{-i\varphi}$

n. Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det U = ad - bc = e^{i\phi}$ (Given)

Let, $V = e^{-i\phi/2}U \Rightarrow V = \begin{pmatrix} e^{-i\phi/2}a & e^{-i\phi/2}b \\ e^{-i\phi/2}c & e^{-i\phi/2}d \end{pmatrix}$

$\det V = e^{-i\phi}ad - e^{-i\phi}bc = e^{-i\phi}(ad - bc) = e^{-i\phi}e^{i\phi} = 1$

We have to multiply U with $e^{-i\phi/2}$ to get V with determinant = 1

Correct answer is (a)

Consider a n x n (n > 1) matrix A, in which A_{ij} is the product of the indices i and j (namely $A_{ij} = ij$) The matrix A [CSIR Dec - 2013]

(a) has one degenerate eigenvalue with degeneracy (n-1)
(b) has two degenerate eigenvalues with degeneracies 2 and (n-2)
(c) has one degenerate eigenvalues with degeneracy n
(d) does not have any degenerate eigenvalues

n. Given : $A_{ij} = ij$. So, for a 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}_{3\times 3}$

Since, the second row and third row is multiple of first row and the same is true for columns, then eigenvalue of the matrix will be Tr(A), 0, 0 i.e. 14, 0, 0
'0' is a degenerate eigenvalue with degeneracy '2'
In general, matrix A (order 'n') has a degenerate eigenvalue with degeneracy 'n–1'.

Correct answer is (a)

Consider the matrix

$$M = \begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$$ [CSIR June - 2014]

The eigenvalues of M are
(a) –5, –2, 7
(b) –7, 0, 7
(c) –4i, 2i, 2i
(d) 2, 3, 6

Soln. Sum of the eigenvalues of the matrix M = trace of the matrix M = 0
Product of the eigenvalues of the matrix M = determinant of the matrix M = 0
Only –7, 0, 7 satisfies both the relation.

Correct answer is (b)

9. The matrices

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfy the commutation relations [CSIR June - 2014]

(a) $[A,B] = B + C, [B,C] = 0, [C,A] = B + C$
(b) $[A,B] = C, [B,C] = A, [C,A] = B$
(c) $[A,B] = B, [B,C] = 0, [C,A] = A$
(d) $[A,B] = C, [B,C] = 0, [C,A] = B$

Soln. The given matrix A, B, C satisfy the relations given in option (d)

Correct answer is (d)

10. The column vector $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ is a simultaneous eigenvector of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ if

(a) $b = 0$ or $a = 0$
(b) $b = a$ or $b = -2a$
(c) $b = 2a$ or $b = -a$
(d) $b = a/2$ or $b = -a/2$

[CSIR Dec - 2014]

Soln. Given: The eigenvector $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ is a simultaneous eigenvector of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Therefore, $A\begin{pmatrix} a \\ b \\ a \end{pmatrix} = \lambda_A \begin{pmatrix} a \\ b \\ a \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ a \end{pmatrix} = \lambda_A \begin{pmatrix} a \\ b \\ a \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ a \end{pmatrix} = \lambda_A \begin{pmatrix} a \\ b \\ a \end{pmatrix} \Rightarrow \lambda_A = 1$

and $B\begin{pmatrix} a \\ b \\ a \end{pmatrix} = \lambda_B \begin{pmatrix} a \\ b \\ a \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ a \end{pmatrix} = \lambda_B \begin{pmatrix} a \\ b \\ a \end{pmatrix} \Rightarrow \begin{pmatrix} a+b \\ 2a \\ a+b \end{pmatrix} = \lambda_B \begin{pmatrix} a \\ b \\ a \end{pmatrix}$

$$\Rightarrow \begin{cases} a+b=\lambda_B a \\ 2a=\lambda_B b \end{cases} \Rightarrow \begin{cases} (1-\lambda_B)a+b=0 \\ 2a-\lambda_B b=0 \end{cases}$$

For the non-trivial solution of a and b from the above equations, we should have

$$\begin{vmatrix} 1-\lambda_B & 1 \\ 2 & -\lambda_B \end{vmatrix}=0 \Rightarrow \lambda_B^2-\lambda_B-2=0 \Rightarrow \lambda_B=2,-1$$

For $\lambda_B=2$, we get $b=a$ and for $\lambda_B=-1$, we get $b=-2a$

Correct option is (b)

GATE Previous Years Question with Solutions

1. For any operator $A, i(A^{+}-A)$ is : **[GATE 2001]**

(a) hermitian (b) anti-hermitian (c) unitary (d) orthogonal

Soln. $\left[i(A^{\dagger}-A)\right]^{\dagger}=(i)^{\dagger}\left[(A^{\dagger})^{\dagger}-(A)^{\dagger}\right]=i(A^{\dagger}-A)$

Correct option is (a)

2. If two matrices A and B can be diagonalized simultaneously, which of the following is true? **[GATE 2002]**

(a) $A^2B=B^2A$ (b) $A^2B^2=B^2A$ (c) $AB=BA$ (d) $AB^2AB=BABA^2$

Soln. The diagonal matrices D_A and D_B corresponding respectively, will commute with each other.

$D_A D_B = D_B D_A$

$\Rightarrow$ $P^{-1}APP^{-1}BP=P^{-1}BPP^{-1}AP$ [same matrix is used for A and B]

$\Rightarrow$ $P^{-1}ABP=P^{-1}BAP \Rightarrow AB=BA$

Correct option is (c)

3. Which one of the following is the inverse of the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$? **[GATE 2002]**

(a) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

Soln. $A^{-1}=\dfrac{adjA}{|A|}=\dfrac{1}{1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Correct option is (c)

4. A 3×3 matrix has eigenvalues 0, 2 + i and 2–i. Which one of the following statements is correct? **[GATE 2003]**

(a) the matrix is hermitian (b) the matrix is unitary

(c) the inverse of the matrix exists (d) the determinant of the matrix is zero

Soln. Determinant of matrix = product of the eigenvalues = 0

Correct option is (d)

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5. A real traceless 4×4 unitary matrix has two eigen values –1 and 1. The other eigenvalues are

(a) zero and +2 (b) –1 and +1 (c) zero and +1 (d) +1 and +1 **[GATE 2004]**

Soln. Sum of the eigenvalues = Trace of the matrix

$\Rightarrow \lambda_1+\lambda_2+\lambda_3+\lambda_4=0 \Rightarrow 1+(-1)+\lambda_3+\lambda_4=0$

$\Rightarrow \lambda_3+\lambda_4=0$

So, $\lambda_3=+1, \lambda_4=-1$

Correct option is (b)

6. The eigenvalues of the matrix $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ are

(a) +1 and +1 (b) zero and +1 (c) zero and +2 (d) –1 and +1

Soln. Eigenvalue equation : $\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix}=0 \Rightarrow (\lambda-1)^2+i^2=0$

$$\Rightarrow \lambda^2-2\lambda=0 \Rightarrow \lambda=0,2$$

Correct option is (c)

7. A real traceless 4×4 unitary matrix has two eigen values –1 and 1. The other eigenvalues are

(a) zero and +2 (b) –1 and +1 (c) zero and +1 (d) +1 and +1 **[GATE 2004]**

Soln. Sum of the eigenvalues = Trace of the matrix

$\Rightarrow \lambda_1+\lambda_2+\lambda_3+\lambda_4=0 \Rightarrow 1+(-1)+\lambda_3+\lambda_4=0$

$\Rightarrow \lambda_3+\lambda_4=0$

So, $\lambda_3=+1, \lambda_4=-1$

Correct option is (b)

8. The eigenvalues of the matrix $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ are **[GATE 2004]**

(a) +1 and +1 (b) zero and +1 (c) zero and +2 (d) –1 and +1

Soln. Eigenvalue equation : $\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix}=0 \Rightarrow (\lambda-1)^2+i^2=0$

$$\Rightarrow \lambda^2-2\lambda=0 \Rightarrow \lambda=0,2$$

Correct option is (c)

9. Eigen value of the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 0 & 2i & 0 \end{pmatrix}$ are **[GATE 2005]**

(a) –2, –1, 1, 2 (b) –1, 1, 0, 2 (c) 1, 0, 2, 3 (d) –1, 1, 0, 3

Soln. Sum of the eigenvalues = Trace of the matrix = 0

Only option (a) satisfies this.

Correct option is (a)

10. A linear transformation T, defined as $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_3 \end{pmatrix}$, transform a vector $\vec{x}$ from a three-dimensional real space to a two-dimensional real space. The transformation matrix T is **[GATE 2006]**

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Soln. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$

Correct option is (a)

Common data for Q. 11 and Q. 12

One of the eigen values of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 5 **[GATE 2006]**

11. The other two eigen values are

(a) 0 and 0 (b) 1 and 1 (c) 1 and –1 (d) –1 and –1

Soln. Sum of the eigenvalues = Trace of the matrix = 5

$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = 5$

$\Rightarrow \quad 5 + \lambda_2 + \lambda_3 = 5 \Rightarrow \lambda_2 + \lambda_3 = 0$

So, $\lambda_2 = 1$ and $\lambda_2 = -1$

Correct option is (c)

12. The normalized eigen vector corresponding to the eigen value 5 is:

(a) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (c) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (d) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Soln. Eigenvalue equation : $(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Putting $\lambda = 5$, we get

$$\Rightarrow \quad \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \quad -3x_1 + 3x_2 = 0$ and $-4x_3 = 0$

Therefore, $X = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$, putting normalization condition $X^{\dagger}X = y \Rightarrow K = \frac{1}{\sqrt{2}}$

So, $X = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Correct option is (d)

13. The eigenvalues of a matrix are $i, -2i$ and $3i$. The matrix is **[GATE 2007]**

(a) unitary (b) anti-unitary (c) hermitian (d) anti-hermitian

Soln. Eigenvalues of an anti-hermitian matrix are either zero or purely imaginary

Correct option is (d)

14. The eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ are **[GATE 2007]**

(a) 6, 1 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (b) 2, 5 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) 6, 1 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (d) 2, 5 and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Soln. $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$; eigenvalue equation : $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 2)(\lambda - 5) - 4 = 0 \Rightarrow \lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 6, 1$$

Now, $(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Putting $\lambda = 6 \Rightarrow \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -x_1 + 4x_2 = 0 \Rightarrow x_1 = 4x_2$

So, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Putting $\lambda = 1 \Rightarrow \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow 4x_1 + 4x_2 = 0 \Rightarrow x_1 = -x_2$

So, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Correct option is (a)

15. Consider a vector $\vec{p}=2\hat{i}+3\hat{j}+2\hat{k}$ in the coordinate system $(\hat{i},\hat{j},\hat{k})$. The axes are rotated anti-clockwise about the Y axis by an angle of 60°. The vector $\vec{p}$ in the rotated coordinate sytem $(\hat{i}',\hat{j}',\hat{k}')$ is [GATE 2007]

(a) $(1-\sqrt{3})\hat{i}'+3\hat{j}'+(1+\sqrt{3})\hat{k}'$ (b) $(1+\sqrt{3})\hat{i}'+3\hat{j}'+(1-\sqrt{3})\hat{k}'$

(c) $(1-\sqrt{3})\hat{i}'+(3+\sqrt{3})\hat{j}'+2\hat{k}'$ (d) $(1-\sqrt{3})\hat{i}'+(3-\sqrt{3})\hat{j}'+2\hat{k}'$

Soln. The rotation matrix corresponding to the anti-clockwise rotation about y-axis by an angle θ will be

$$R_y(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix}x'\\ y'\\ z'\end{bmatrix}=\begin{bmatrix}\cos 60^\circ & 0 & -\sin 60^\circ\\ 0 & 1 & 0\\ \sin 60^\circ & 0 & \cos 60^\circ\end{bmatrix}\begin{bmatrix}2\\ 3\\ 2\end{bmatrix}=\begin{bmatrix}(1-\sqrt{3})\\ 3\\ (1+\sqrt{3})\end{bmatrix}$$

Correct option is (a)

16. For arbitrary matrices E, F, G and H, if $EF-FE=0$, then Trace ($EFGH$) is equal [GATE 2008]

(a) Trace (HFEG)
(b) Trace (E), Tace (F), Trace (G), Trace (H)
(c) Trace (GFEH)
(d) Trace (EGHF)

Soln. $Tr(EFGH)=Tr(HEFG)=Tr(HFEG)$

Correct option is (a)

17. An unitary matrix $\begin{bmatrix}ae^{i\alpha} & b\\ ce^{i\beta} & d\end{bmatrix}$ is given, where a, b, c, d, α and β are real. The inverse of the matrix is [GATE 2008]

(a) $\begin{bmatrix}ae^{i\alpha} & -ce^{i\beta}\\ b & d\end{bmatrix}$ (b) $\begin{bmatrix}ae^{i\alpha} & ce^{i\beta}\\ b & d\end{bmatrix}$ (c) $\begin{bmatrix}ae^{i\alpha} & b\\ ce^{i\beta} & d\end{bmatrix}$ (d) $\begin{bmatrix}ae^{-i\alpha} & ce^{-i\beta}\\ b & d\end{bmatrix}$

Soln. For unitary matrix, $U^{-1}=U^{+}=\begin{bmatrix}a\,e^{-i\alpha} & c\,e^{-i\beta}\\ b & d\end{bmatrix}$

Correct option is (d)

18. The eigenvalues of the matrix $A=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$ are [GATE 2009]

(a) real and distinct (b) complex and distinct
(c) complex and coinciding (d) real and coinciding

Soln. Eigenvalue equation : $\begin{vmatrix}-\lambda & i\\ i & -\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-i^2=0 \Rightarrow \lambda=\pm i$

Correct option is (b)

19. The eigen values of the matrix $\begin{pmatrix}2 & 3 & 0\\ 3 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$ are [GATE 2010]

(a) 5, 2, –2 (b) –5, –1, 1 (c) 5, 1, –1 (d) –5, 1, 1

Soln. Sum of the eigenvalues = Trace of the matrix

i.e. $\lambda_1+\lambda_2+\lambda_3=5$

Product of the eigenvalues = Determinant of the matrix

i.e. $\lambda_1\lambda_2\lambda_3=-5$

Only option (c) satisfy both relations.

Correct option is (c)

20. Two matrices A and B are said to be similar if $B=P^{-1}AP$ for some invertible matrix P. Which of the following statements is **NOT TRUE** ? [GATE-2011]

(a) Det A = Det B (b) Trace of A = Trace of B
(c) A and B have the same eigenvectors (d) A and B have the same eigenvalues

Soln. Two similar matrices A and B have same trace, determinant eigenvalues and rank.

Correct option is (c)

21. A 3 × 3 matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to be positive integers. The largest eigenvalue of the matrix is : [GATE-2011]

(a) 18 (b) 12 (c) 9 (d) 6

Soln. Trace of the matrix $=\lambda_1+\lambda_2+\lambda_3=11$

Determinant of the matrix $=\lambda_1\lambda_2\lambda_3=36$

Combination of λ_1, λ_2 and λ_3 that will satisfy both relation will be $\lambda_1=6, \lambda_2=3, \lambda_3=2$.

Correct option is (d)

22. The eigenvalues of the matrix $\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ are [GATE-2012]

(a) 0, 1, 1 (b) $0, -\sqrt{2}, \sqrt{2}$ (c) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$ (d) $\sqrt{2}, \sqrt{2}, 0$

Soln. Trace of the matrix $=\lambda_1+\lambda_2+\lambda_3=0$

Only option (b) satisfy this.

Correct option is (b)

23. The degenerate eigenvalues of the matrix $M=\begin{pmatrix}4 & -1 & -1\\ -1 & 4 & -1\\ -1 & -1 & 4\end{pmatrix}$ is:......... [GATE-2013]

(Your answer should be an integer)

Soln. Trace of the matrix $=\lambda_1+\lambda_2+\lambda_3=12$

Determinant of the matrix $= \lambda_1 \lambda_2 \lambda_3 = 50$

The combination $\lambda_1 = 5, \lambda_2 = 5, \lambda_3 = 2$ satisfies the above two relations.

Correct answer is (5)

24. The matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is **[GATE-2014]**

(a) orthogonal (b) symmetric (c) anti-symmetric (d) unitary

Soln. $AA^{+} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So, A is an unitary matrix

Correct option is (d)

IIT-JAM Previous Year Questions with Solutions

1. Which of the following is INCORRECT for the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$?

(a) It is its own inverse
(b) It is its own transpose
(c) It is non-orthogonal
(d) It has eigen values ±1.

Soln. $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow MM^T = M^T M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ and $|M| = -1$.

So, M is orthogonal.

Correct option is (c)

2. The symmetric pair of $P = \begin{pmatrix} a \\ b \end{pmatrix}(a-2\ \ b)$ is:

(a) $\begin{pmatrix} a^2-2 & ba-1 \\ ba-1 & b^2-2 \end{pmatrix}$ (b) $\begin{pmatrix} a(a-1) & b \\ b & b^2 \end{pmatrix}$ (c) $\begin{pmatrix} a(a-1) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$ (d) $\begin{pmatrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$

Soln. $P = \begin{pmatrix} a \\ b \end{pmatrix}(a-2\ \ b) = \begin{pmatrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$

Symmetrical part of $P = \frac{1}{2}\left(P + P^T\right) = \begin{pmatrix} a(a-2) & b(a-1) \\ b(a-1) & b^2 \end{pmatrix}$

Correct option is (d)

3. $(x\ \ y)\begin{pmatrix} 5 & -7 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 15$

The matrix equation above represents.

(a) A circle of radius $\sqrt{15}$
(b) An ellipse of semi major axis $\sqrt{5}$
(c) An ellipse of semi major axis 5
(d) A hyperbola.

Soln. $(x\ \ y)\begin{pmatrix} 5 & -7 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 15 \Rightarrow 5x^2 + 3y^2 = 15 \Rightarrow \frac{x^2}{3} + \frac{y^2}{5} = 1$

It is a equation of ellipse with semi-major axis $\sqrt{5}$ units.

Correct option is (b)

4. The product PQ of any two real, symmetric matrices P and Q is:

(a) Symmetric for all P and Q
(b) Never symmetric
(c) Symmetric if $PQ = QP$
(d) Antisymmetric for all P and Q

Soln. $P^T = P, Q^T = Q;\ (PQ)^T = Q^T P^T = QP$

For PQ to be symmetric, $PQ = QP$ should be satisfied.

Correct option is (c)

5. A matrix is given by $M = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$. The eigenvalues of the M are

(a) Real and positive
(b) Purely imaginary with modulus 1
(c) Complex with modulus 1
(d) Real and negative.

Soln. $M = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$

Eigenvalue equation : $\begin{vmatrix} \frac{i}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} - \lambda \end{vmatrix} = 0$

$$\Rightarrow \left(\lambda - \frac{i}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0 \Rightarrow \lambda = \frac{i}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}$$

Both $|\lambda_1| = 1, |\lambda_2| = 1$

Correct option is (c)

6. Given two $(n \times n)$ matrices $\hat{P}$ and $\hat{Q}$ such that $\hat{p}$ is hermitian and $\hat{Q}$ is skew (anti)-hermitian. Which one of the following combintations of $\hat{P}$ and $\hat{Q}$ is necessarily a Hermitian matrix?

(a) $\hat{P}\hat{Q}$ (b) $i\hat{P}\hat{Q}$ (c) $\hat{P} + i\hat{Q}$ (d) $\hat{P} + \hat{Q}$

Soln. $\left(\hat{P}\right)^\dagger = \hat{P}, \left(\hat{Q}\right)^\dagger = -\hat{Q} \Rightarrow \left(\hat{P} + i\hat{Q}\right)^\dagger = \hat{P} + (-i)\left(-\hat{Q}\right) = \hat{P} + i\hat{Q}$

Correct option is (c)

7. The inverse of the matrix $M=\begin{pmatrix}0&1&1\\0&0&1\\1&0&0\end{pmatrix}$ is

(a) $M-I$ (b) M^2-I (c) $1-M^2$ (d) $1-M$

where I is the identity matrix.

Soln. Eigenvalue equation : $|M-\lambda I|=0 \Rightarrow \begin{vmatrix}-\lambda&1&1\\0&-\lambda&1\\1&0&-\lambda\end{vmatrix}=0$

$\Rightarrow -\lambda\left(\lambda^2\right)-1(-1)+1(\lambda)=0 \Rightarrow \lambda^3-\lambda-1=0$

According to Cayley hamilton theorem, $M^3-M-I=0$

$\Rightarrow M^3-M=I \qquad \Rightarrow M^{-1}=M^2-I$

Correct option is (b)

8. (a) Find the normalized eigenvector α of the matrix $M=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, corresponding to its positive eigenvalue.

(b) The normalized eigenvectors of the matrix $N=\begin{bmatrix}0&-i\\i&0\end{bmatrix}$ are β_1 and β_2 with the eigenvalues λ_1 and λ_2 respectively and $\lambda_1>\lambda_2$. If the eigenvector α obtained in part (a) is expressed as $\alpha=P\beta_1+Q\beta_2$. Find the constants P and Q.

Soln. (a) $M=\begin{bmatrix}0&1\\1&0\end{bmatrix} \Rightarrow$ eigenvalue equation : $|M-\lambda I|=\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix}=0$

$\Rightarrow \quad \lambda^2-1=0 \qquad \Rightarrow \lambda=1,-1$

Suppose, the eigenvector corresponding $\lambda=1$ is $X=\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$[M-\lambda I]X=0 \qquad \Rightarrow \begin{bmatrix}-\lambda&1\\1&-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

Putting $\lambda=1$, we get $\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow -x_1+x_2=0$

$\Rightarrow \quad x_1=x_2;\ say\ x_1=x_2=k \Rightarrow x=\begin{bmatrix}k\\k\end{bmatrix}$

For normalized eigenvector, $X^\dagger X=1 \Rightarrow [k^*k^*]\begin{bmatrix}k\\k\end{bmatrix}=1$

$\Rightarrow \quad 2|k|^2=1 \Rightarrow |k|=\frac{1}{\sqrt{2}} \Rightarrow x=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

Therefore, $\alpha=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

(b) $N=\begin{bmatrix}0&-i\\i&0\end{bmatrix} \Rightarrow$ Eigenvalue equation : $|N-\lambda I|=0 \Rightarrow \begin{vmatrix}-\lambda&-i\\i&-\lambda\end{vmatrix}=0$

$\Rightarrow \quad \lambda^2-1=0 \Rightarrow \lambda=1,-1$

Similarly as before, normalised eigenvectors corresponds to $\lambda_1=1, \lambda_2=-1$ are

$\beta_1=\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\1\end{bmatrix}$ and $\beta_2=\frac{1}{\sqrt{2}}\begin{bmatrix}i\\1\end{bmatrix}$

According to the question, $\alpha=P\beta_1+Q\beta_2$

$\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}=P.\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\1\end{bmatrix}+Q.\frac{1}{\sqrt{2}}\begin{bmatrix}i\\1\end{bmatrix}$

$\Rightarrow \quad P-Q=i; \qquad P+Q=1$

Solving we get, $P=\frac{1+i}{2}, Q=\frac{1-i}{2}$

9. The trace of a 2×2 matrix is 4 and its determinant is 8. If one of the eigenvalue is $2(1+i)$, the other eigenvalue is

(a) $2(1-i)$ (b) $2(1+i)$ (c) $(1+2i)$ (d) $(1-2i)$

Soln. Sum of the eigenvalues = Trace of the matrix

$\Rightarrow \quad \lambda_1+\lambda_2=4 \quad \Rightarrow \quad (2+2i)+\lambda_2=4$

$\Rightarrow \quad \lambda_2=2-2i=2(1-i)$

Correct option is (a)

PRACTICE SET

1. Which of the following is true for the matrix $\begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$?

(a) It is an idempotent matrix
(b) It is nilpotent matrix of order 3
(c) It is an involutory matrix
(d) None of the above

2. Which of the following statements is true for the matrix $\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$?

(a) It is a idempotent matrix
(b) It is a nilpotent matrix of order 2
(c) It is an involutary matrix
(d) It is a periodic matrix of order 2

3. The product of the two matrices $\begin{pmatrix} \cos^2\theta & \cos\theta.\sin\theta \\ \cos\theta.\sin\theta & \sin^2\theta \end{pmatrix}$ and $\begin{pmatrix} \cos^2\phi & \cos\phi.\sin\phi \\ \cos\phi.\sin\phi & \sin^2\phi \end{pmatrix}$ is zero.
Which of the following statements is true ?

(a) $\theta-\phi$ is odd multiple of $(\pi/2)$
(b) $\theta-\phi$ is even multiple of $(\pi/2)$
(c) $\theta-\phi$ is odd multiple of π
(d) $\theta-\phi$ is even multiple of π

4. To convert a Hermitian matrix into skew-Hermitian one, the Hermitian matrix must be multiplied by
(a) – 1
(b) i
(c) – i
(d) None of the above

5. Suppose the matrix $B^{\dagger}AB$ is Hermitian in nature. Which of the following statement is true ?
(a) A is a hermitian matrix
(b) B is a hermitian matrix
(c) A is a skew hermitian matrix
(d) B is a skew hermitian matrix

6. Which of the following statements is true?

(a) Product of two orthogonal matrices A and B will be orthogonal if A and B commutes

(b) Product of two orthogonal matrices A and B will be always orthogonal

(c) Product of two orthogonal matrices A and B will be never orthogonal

(d) None of these

7. The matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is

(a) Symmetric
(b) Skew Hermitian
(c) Orthogonal
(d) Unitary

8. Suppose A is $n \times n$ square invertible matrix. Which of the following statements is true for A (adj A) ?
(a) It is a null matrix of order 'n'
(b) It is an unit matrix of order 'n'
(c) It is a scalar matrix of order 'n'
(d) None of the these

9. If 'A' is a 3×3 matrix with determinant 2, then the determinant of $adj.[adj.[adj.(A^{-1})]]$ is equal to

(a) $\frac{1}{512}$
(b) $\frac{1}{1024}$
(c) $\frac{1}{128}$
(d) $\frac{1}{256}$

10. For which value of λ, the following matrix $A = \begin{bmatrix} 0 & -4 & 1 \\ 2 & \lambda & -3 \\ 1 & 2 & -1 \end{bmatrix}$ will be invertible in nature ?

(a) $\lambda = 4$
(b) $\lambda \neq 8$
(c) $\lambda \neq 4$
(d) none

11. If 'U' is unit matrix and the matrix 'A' is defined as $A = 1 - 2UU^T$, then A^{-1} is equal to

(a) $1-2UU^T$
(b) $1+2UU^T$
(c) $2UU^T - 1$
(d) $4UU^T$

12. If a square matrix A has an eigenvalue λ, then the eigenvalue of the matrix $(kA)^T$ (where $k \neq 0$) is

(a) λ/k
(b) k/λ
(c) λk
(d) None of these

13. The eignvalues of the matrix. $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are

(a) 1, 3, 1
(b) 2, 3, 2
(c) 5, 1, 1
(d) 3, 5, 2

14. Two eigen values of the matrix $\begin{bmatrix} 1 & 0 & 0 & -\frac{a}{2} \\ 0 & 1 & 0 & -\frac{a}{2} \\ 0 & 0 & 1 & -\frac{a}{2} \\ 0 & 0 & 0 & a \end{bmatrix}$ are

(a) 1 and a
(b) 1 and – a
(c) $\frac{1}{2}$ and $-\frac{a}{2}$
(d) $\frac{1}{2}$ and $\frac{a}{2}$

15. The trace of the matrix of a 2 × 2 matrix is 1 and determinant 1. Which of the following has to be true?

(a) one of the eigenvalues is 0
(b) one of the eigenvalue is 1
(c) both of the eigenvalues are 1
(d) neither of the eigenvalues are 1

16. For which of the following matrices both eigenvalues are positive?

(a) $\begin{pmatrix} -7 & 2 \\ 2 & -7 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
(c) $\begin{pmatrix} 4 & -3 \\ -3 & 5 \end{pmatrix}$
(d) $\begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}$

17. For an N X N matrix consisting of all ones,

(a) all eigenvalues are 1
(b) all eigenvalues are 0
(c) eigenvalues are 1,2,3,...... N
(d) one eigenvalue is N and others are 0

18. A 2×2 matrix 'A' has eigenvalues $e^{i\pi/4}$ and $e^{i\pi/5}$ such that $A^n = I$. The smallest possible value of 'n' is

(a) 20 (b) 30 (c) 40 (d) 80

19. Let A be a 3 x 3 hermitian matrix which satisfies the matrix equation $M^2 - 7M + 12I = 0$, where I refers to the identity matrix. What is the determinant of the matrix M given that the trace is 10?

(a) 27 (b) 36 (c) 48 (d) 64

20. The eigenvalues of the matrix $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ are

(a) $\{1,1,1,1\}$ (b) $\{1,1,-1,-1\}$ (c) $\{0,0,\sqrt{2},-\sqrt{2}\}$ (d) $\{\sqrt{2},-\sqrt{2},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\}$

21. The eigen values of $(A^4 + 3A - 2I)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, are

(a) 2, 20, 88 (b) 1, 2, 3 (c) 2, 20, 3 (d) 1, 20, 88

22. If a matrix A is such that $3A^3 + 2A^2 + 5A + I = 0$, then A^{-1} is equal to

(a) $-(3A^2 + 2A + 5)$ (b) $3A^2 + 2A + 5$ (c) $3A^2 - 5A - 5$ (d) None of these

23. Let, 'P' be a 3 x 3 Hermitian matrix which satisfies the matrix equation $M^2 - 5M + 6I = 0$. Which of the following are the possible eigenvalues of M?

(a) (0,1,2) (b) (0,2,3) (c) (1,3,4) (d) (2,3,4)

24. The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ can be related by a similarity transformation to a matrix

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

25. Two different 2×2 matrices A and B found to be have same eigenvalues. It is then correct to state that $A = SBS^{-1}$ where S can be a

(a) arbitary 2×2 matrix (b) hermitian 2×2 matrix

(c) traceless 2×2 matrix (d) unitary 2×2 matrix

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26. The rank of the matrix $\begin{pmatrix} \Omega & -1 & 0 \\ 0 & \Omega & -1 \\ -1 & 0 & \Omega \end{pmatrix}$ is 2, for Ω equal to

(a) 0 (b) 1 (c) 2 (d) 4

27. The values of a and b for which the following systems of linear equation.

$$ax + y + 3z = a$$
$$2x + by - z = 3$$
$$5x + 7y + z = 7$$

has an infinite number of solutions, are

(a) $a = 1, b = 1$ (b) $a = 1, b = 3$ (c) $a = 2, b = 3$ (d) $a = 2, b = 1$

28. The value of 'a' for which the system of equations

$$a^3x + (a+1)^3 y + (a+2)^3 z = 0$$
$$ax + (a+1)y + (a+2)z = 0$$
$$x + y + z = 0$$

has a non-zero solution, is equal to

(a) 1 (b) 0 (c) -1 (d) None of these

29. Find the eigenvalues and normalized eigen vectors of the following matrices:

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

30. Find the characteristic equation, of the following matrix and verify Cayley-Hamilton theorem.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

31. Show that, $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ satisfies his own eigen equation and hence find A^{-1}.

32. If α is the characteristic root of a non-singular matrix A, then show that, $\frac{|A|}{\alpha}$ is a characteristic root o (adj. A).

33. Find the Characteristic equation of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ and hence find

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

34. The matrix $A = \begin{pmatrix} p & r \\ r & q \end{pmatrix}$ is transformed to the diagonal form $D = M^{-1}AM$, where

$M=\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}$. Find the value of ϕ which gives the diagonal transformation.

ANSWER KEY

1. (c) 2. (b) 3. (a) 4. (b) 5. (a) 6. (b) 7. (d)
8. (c) 9. (c) 10. (b) 11. (a) 12. (c) 13. (c) 14. (a)
15. (d) 16. (c) 17. (d) 18. (c) 19. (b) 20. (b) 21. (a)
22. (d) 23. (b) 24. (d) 25. (d) 26. (b) 27. (b) 28. (c)

Chapter 3

DIFFERENTIAL EQUATIONS

3.1 Basic Review of Differential Equation

Definition:

An equation which involves differential co-efficient is called differential equation.

Example: $\frac{dy}{dx}=\frac{1-x^2}{1+y^2}$; $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+6y=0$

All differential equations can be classified into two types:

(i) Ordinary differential equation: This consists of derivatives w.r.t. a single independent variable.

(ii) Partial differential equation: This consists of partial derivatives w.r.t. more than one independent variable.

Order and Degree of a differential equation:

The order of the highest order derivative present in the given equation is known as order of the differential equation and the power of the highest order derivative term present in the given equation is known as degree of differential equation.

Example: $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E\sin\omega t$ ⇒ order 2, degree 1

$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$ ⇒ order 2, degree 2

Linear Differential Equation:

A differential equation is said to be linear if (i) every dependent variable and its derivative occurs in first degree only and (ii) no products of dependent variables and its derivatives occur.

Example: $\frac{dy}{dx}=(x+\sin x)$, $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+7y=0$

Family of Curves:

An n-parameter family of curves is a set of relations of the form

$$\{(x,y):f(x,y,c_1,c_2\text{.........}c_n)\}=0$$

where 'f' is a real valued function of $x,y,c_1,c_2\text{.........}c_n$ and each c_i ranges over an interval of real values.

Example: The set of concentric circles is defined by $x^2+y^2=c$ is a one parameter family if c takes all positive real values.

There will be n arbitary constants in n-parameter family of curves. Then, we can obtain n^{th} order differential equation whose solution is the given family.

Example 1. The differential equation of all circles passing through the origin and having their centres on the y-axis

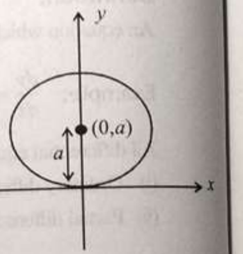
(a) $(x^2+y^2)dy/dx=2xy$ (b) $(x^2-y^2)dy/dx=2xy$

(c) $dy/dx=2xy(x^2+y^2)$ (d) $dy/dx=2xy(x^2-y^2)$

Soln: Equation of the circle passing through the origin and having their centers on the y-axis will be of the form

$$x^2+(y-a)^2=a^2$$

$$\Rightarrow 2x+2(y-a)\frac{dy}{dx}=0 \Rightarrow (y-a)=-\frac{x}{(dy/dx)} \Rightarrow a=y+\frac{x}{(dy/dx)}$$



Therefore, $x^2+\dfrac{x^2}{(dy/dx)^2}=\left(y+\dfrac{x}{(dy/dx)}\right)^2$

$$\Rightarrow x^2=y^2+\frac{2xy}{(dy/dx)} \Rightarrow (x^2-y^2)\frac{dy}{dx}=2xy$$

Correct option is (b)

Example 2: The function $y=\ln[sin(x+a)]+b$ is the solution of the following differential equation:

(a) $y''=-\left[1+(y')^2\right]$ (b) $y''=y^2-(y')^2$ (c) $y''=\left[1+(y')^2\right]$ (d) $y''=y'+y^2$

Soln: $y=\ln[\sin(x+a)]+b \Rightarrow \dfrac{dy}{dx}=\cot(x+a) \Rightarrow \dfrac{d^2y}{dx^2}=-\operatorname{cosec}^2(x+a)=-\left[1+\cot^2(x+a)\right]$

Therefore, $\dfrac{d^2y}{dx^2}=-\left[1+\left(\dfrac{dy}{dx}\right)^2\right]$

Correct option is (a)

3.2 Differential Equations of the First Order and First Degree:

CASE I: Separation of Variables

Every Differential equation can be written in the form,

$$f(y)dy=\phi(x)dx \Rightarrow \int f(y)dy=\int \phi(x)dx+C$$

Example 3: Solve $e^{dy/dx}=(x+1)$; given $y=3$ at $x=0$

Soln: Taking log of both sides we get, $\dfrac{dy}{dx}=\ln(x+1) \Rightarrow dy=\ln(x+1)dx$

on integration, $\int dy=\int 1.\ln(x+1)dx$

$$\Rightarrow \quad y=x\ln(x+1)-\int\frac{x}{(x+1)}dx+C \quad \Rightarrow \quad y=x\ln(x+1)-\int\frac{(x+1)-1}{(x+1)}dx+C$$

$$\Rightarrow \quad y=x\ln(x+1)-\int\frac{(x+1)}{(x+1)}dx+\int\frac{1}{(x+1)}dx+C \quad =x\ln(x+1)-x+\ln(x+1)+C$$

$$\Rightarrow \quad y=(x+1)\ln(x+1)-x+C$$

Given: at $x=0$ $y=3$ $\Rightarrow$ $C=3$

Therefore, $y=(x+1)\ln(x+1)-x+3$

Example 4: Solve $\dfrac{dy}{dx}=e^y\left(e^x+x^2e^{x^3}\right)$

Soln: $\dfrac{dy}{dx}=e^y\left(e^x+x^2e^{x^3}\right) \Rightarrow \int e^{-y}dy=\int\left(e^x+x^2e^{x^3}\right)dx \Rightarrow -e^{-y}=e^x+(1/3)e^{x^3}+C$

Example 5: Solve $(x+y)(dx-dy)=dx+dy$

Soln: Re-writting the given equation, we get $(x+y-1)dx=(x+y+1)dy \Rightarrow \dfrac{dy}{dx}=\dfrac{x+y-1}{x+y+1}$

Let $x+y=v \Rightarrow \dfrac{dy}{dx}=(dv/dx)-1$

Therfore the given equation becomes,

$$\frac{dv}{dx}-1=\frac{v-1}{v+1} \Rightarrow \frac{dv}{dx}=\frac{2v}{v+1} \Rightarrow \int 2dx=\int\left(1+\frac{1}{v}\right)dv$$

$$\Rightarrow 2x+c=v+\ln v \quad \Rightarrow -y+c=\ln(x+y)$$

Example 6: Solve $\dfrac{dy}{dx}=\dfrac{4x+6y+5}{3y+2x+4}$

Soln: The given equation may be re-written as $\dfrac{dy}{dx}=\dfrac{2(2x+3y)+5}{(2x+3y)+4}$

Taking $2x+3y=v \Rightarrow 2+3\dfrac{dy}{dx}=\dfrac{dv}{dx} \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{3}\left(\dfrac{dv}{dx}-2\right)$

Therefore, the given equation becomes,

$$\frac{1}{3}\left(\frac{dv}{dx}-2\right)=\frac{2v+5}{v+4} \Rightarrow \frac{dv}{dx}=\frac{3(2v+5)}{v+4}+2=\frac{8v+23}{v+4}$$

$$\Rightarrow \frac{dx}{dv}=\frac{v+4}{8v+23}=\frac{\frac{1}{8}(8v+23)+\left(4-\frac{23}{8}\right)}{8v+23}=\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right] \Rightarrow \int dx=\int\left[\frac{1}{8}+\frac{9}{8(8v+23)}\right]dv$$

$$\Rightarrow x+c=\left(\frac{v}{8}\right)+\left(\frac{9}{64}\right)\log(16x+24y+23) \quad \Rightarrow 3y-6x+\left(\frac{9}{8}\right)\log(16x+24y+23)=8c$$

$\Rightarrow y-2x+\left(\frac{3}{8}\right)\log(16x+24y+23)=c'$ where $c'(=8c/3)$ is an arbitrary constant.

CASE II: Homogeneous Differential equation:

A differential equation $\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$ is said to homogeneous if f(x, y) and ϕ(x, y) are of same degree.

In such equations, put $y=vx$ & $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Example 7: Solve $xdy-ydx=\sqrt{(x^2+y^2)}dx$

Soln: $\frac{dy}{dx}=\frac{y+\sqrt{(x^2+y^2)}}{x}=\frac{y}{x}+\sqrt{\left\{1+\left(\frac{y}{x}\right)^2\right\}}$

Take, $\frac{y}{x}=v$, i.e. $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow v+x\frac{dv}{dx}=v+\sqrt{(1+v^2)} \Rightarrow \int\frac{dx}{x}=\int\frac{dv}{\sqrt{(1+v^2)}} \quad \Rightarrow \log x+\log c=\log\left[v+\sqrt{(v^2+1)}\right]$$

$$\Rightarrow xc=v+\sqrt{(v^2+1)} \quad \Rightarrow x^2c=y+\sqrt{(y^2+x^2)}$$

Example 8: Solve $x\cos\left(\frac{y}{x}\right)(ydx+xdy)=y\sin\left(\frac{y}{x}\right)(xdy-ydx)$

Soln: $\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} \quad \Rightarrow \frac{dy}{dx}=\frac{\left[\cos\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)\right]\left(\frac{y}{x}\right)}{\left[\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)-\cos\left(\frac{y}{x}\right)\right]}$

Take, $\frac{y}{x}=v$ i.e. $y=vx \quad \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v} \quad \Rightarrow x\frac{dv}{dx}=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v}-v=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow \int 2\frac{dx}{x}=\int\frac{v\sin v-\cos v}{v\cos v}dv=\int\left[\frac{\sin v}{\cos v}-\frac{1}{v}\right]dv \Rightarrow 2\log x=-\log\cos v-\log v+\log c$$

$$\Rightarrow \log x^2=\log(c/v\cos v) \Rightarrow xy\cos\frac{y}{x}=c$$

Example 9: Solve $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Soln: Take $x=X+h,\ y=Y+k$, so that $\frac{dy}{dx}=\frac{dY}{dX}$ ...(1)

Therefore, given equation becomes $\frac{dy}{dx}=\frac{X+2Y+(h+2k-3)}{2X+Y+3+(2h+k-3)}$ ...(2)

Choose h, k so that $h+2k-3=0$ and $2h+k-3=0$ ...(3)

Solving (3) we get h = 1, k = 1 so that from (1), we have $X=x-1, Y=y-1$

Using (3) in (2), we get $\frac{dY}{dX}=\frac{X+2Y}{2X+Y}=\frac{1+(2Y/X)}{2+(Y/X)}$

Take $\frac{Y}{X}=v$, i.e. $Y=vX$. $\Rightarrow \frac{dY}{dX}=v+X\frac{dv}{dX}$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{1+2v}{2+v} \Rightarrow X\frac{dv}{dX}=\frac{1+2v}{2+v}-v=\frac{1-v^2}{2+v}$$

$\Rightarrow \int\frac{dX}{X}=\int\frac{(2+v)dv}{(1-v)(1+v)}=\int\left[\frac{1}{2}\left(\frac{1}{1+v}\right)+\frac{3}{2}\left(\frac{1}{1-v}\right)\right]dv$ [Resolving into partial fractions]

$$\Rightarrow \log X+\log c=\left(\frac{1}{2}\right)\left[\log(1+v)-3\log(1-v)\right] \Rightarrow 2\log(cX)=\log\frac{1+v}{(1-v)^3} \Rightarrow X^2c^2=\frac{1+v}{(1-v)^3}$$

$$\Rightarrow X^2c^2\left(1-\frac{Y}{X}\right)^3=1+\frac{Y}{X} \Rightarrow c^2(X-Y)^3=X+Y \Rightarrow c^2\{x-y\}^2=x+y-2$$

CASE III: Linear differential equations of first order:

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are the functions of x only or constant.

For the solution of these equations we have to calculate integral factor $(I.F)=e^{\int P(x)dx}$ and the solution of differential equation will be of the form

$$I.F.\times y=\int Q(x)\times I.F.dx+C$$

Example 10: Solve $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$

Soln: The equation can be rewritten as $\frac{dy}{dx}-\frac{y}{x+1}=e^x(x+1)$

Compare with the standard form of first order linear differential equation we get,

$$P=-\frac{1}{x+1} \text{ and } Q=e^x(x+1)$$

Integrating factor, $I.F.=e^{\int Pdx}=e^{-\int\frac{dx}{x+1}}=e^{-\log(x+1)}=\frac{1}{x+1}$

Therefore, the solution : $y\frac{1}{x+1}=\int e^x(x+1)\frac{1}{(x+1)}dx=\int e^x dx \Rightarrow y=e^x(x+1)+C$

Example 11: Solve $(1-x^2)\left(\frac{dy}{dx}\right)+2xy=x\sqrt{(1-x^2)}$

Soln: The given equation is $\frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x}{\sqrt{(1-x^2)}}$

I.F. $=e^{\int Pdx}=e^{\int\frac{2x}{1-x^2}dx}=e^{-\ln(1-x^2)}=\frac{1}{1-x^2}$

So the required solution is

$$\frac{y}{1-x^2}=\int\frac{x}{\sqrt{(1-x^2)}}\times\frac{1}{1-x^2}dx=\int\frac{x}{(1-x^2)^{3/2}}dx$$

Putting $1-x^2=t$ and $-2xdx=dt$, we get

$$\Rightarrow\frac{y}{1-x^2}=-\frac{1}{2}\int t^{-3/2}dt=t^{-1/2}+c$$

The solution is $\frac{y}{1-x^2}=\frac{1}{\sqrt{(1-x^2)}}+c$

Example 12: Integrate $(1+x^2)\left(\frac{dy}{dx}\right)+2xy-4x^2=0$. Obtain equation of the curve satisfying this equation and passing through origin.

Soln: Re-writting the given equation, $\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$

Here, I.F. $=e^{\int Pdx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\ln(1+x^2)}=(1+x^2)$

Hence the requied solution,

$$y(1+x^2)=\int\frac{4x^2}{1+x^2}(1+x^2)dx+c \Rightarrow y(1+x^2)=\left(\frac{4}{3}\right)x^3+c$$

Since the required curve passes through origin i.e. for $x=0, y=0 \Rightarrow c=0$

Hence, the equation of the required curve is $4x^3=3y(1+x^2)$

Example 13: Solve $\frac{dy}{dx}+\frac{y}{(1-x^2)^{3/2}}=\frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$

Soln: Comparing the given equation with $\left(\frac{dy}{dx}\right)+Py=Q$, here $P=\frac{1}{(1-x^2)^{3/2}}$ and $Q=\frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$

Here I.F. $=e^{\int Pdx}=e^{\int\frac{1}{(1-x^2)^{3/2}}dx}$ $[Putting\ x=\sin\theta\Rightarrow dx=\cos\theta d\theta]=e^{\int\sec^2\theta d\theta}=e^{\tan\theta}=e^{\frac{x}{\sqrt{1-x^2}}}$

Solution of the given differential equation is

$$y.e^{\frac{x}{\sqrt{1-x^2}}}=\int\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}.e^{\frac{x}{\sqrt{1-x^2}}}dx=\int\frac{x/\sqrt{(1-x^2)}+1}{(1-x^2)^{3/2}}e^{x/(1-x^2)^{1/2}}dx \qquad \ldots(1)$$

Putting $\frac{x}{\sqrt{1-x^2}}=t\Rightarrow\frac{\sqrt{(1-x^2)}.1-x\left(\frac{1}{2}\right)(1-x^2)^{-1/2}.(-2x)}{1-x^2}dx=dt$

$$\Rightarrow\frac{\sqrt{(1-x^2)}+\left[\frac{x^2}{\sqrt{(1-x^2)}}\right]}{1-x^2}dx=dt\Rightarrow\frac{1}{(1-x^2)^{3/2}}dx=dt$$

From equation (1) we get :

$$y.e^{\frac{x}{\sqrt{1-x^2}}}=\int(t+1)e^t dt=(t+1)e^t-\int e^t dt=(t+1)e^t-e^t=te^t$$

$$\Rightarrow ye^{x/(1-x^2)^{1/2}}=\frac{x}{\sqrt{(1-x^2)}}e^{x/(1-x^2)^{1/2}}+c\Rightarrow y=\frac{x}{\sqrt{(1-x^2)}}+c\,e^{-x/(1-x^2)^{1/2}}$$

Example 14: Solve $x(1-x^2)dy+(2x^2y-y-ax^3)dx=0$

Soln: Re-writting the given equation, we have

$$x(1-x^2)\frac{dy}{dx}+y(2x^2-1)=ax^3 \ or\ \frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}y=\frac{ax^2}{1-x^2}$$

Compare this equation with $\left(\frac{dy}{dx}\right)+Py=Q$, we have $P=\frac{2x^2-1}{x(1-x^2)}$ and $Q=\frac{ax^2}{1-x^2}$

Therefore, integrating factor $=e^{\int Pdx}=e^{\int\frac{2x^2-1}{x(1-x^2)}dx}$

Now, $\int\frac{2x^2-1}{x(1-x^2)}dx=-\int\left[\frac{1}{x}+\frac{1}{2(x+1)}+\frac{1}{2(x-1)}\right]dx=-\left[\ln x+\frac{1}{2}\ln(x+1)+\frac{1}{2}\ln(x-1)\right]$

$$= -\ln\left[x\sqrt{(x^2-1)}\right] = \ln\left[x\sqrt{(x^2-1)}\right]^{-1}$$

Therefore, I. F. $= \dfrac{1}{x\sqrt{x^2-1}}$

Hence, the solution will be

$$\Rightarrow \frac{y}{x\sqrt{(x^2-1)}} = \int \frac{ax^2}{1-x^2} \times \frac{1}{x\sqrt{(x^2-1)}}dx + c = c - a\int \frac{xdx}{(x^2-1)^{3/2}} = c - \frac{a}{2}\int \frac{dt}{t^{3/2}}, \text{ [putting } x^2-1=t\text{]}$$

$$\Rightarrow \frac{y}{x\sqrt{(x^2-1)}} = c + \frac{a}{\sqrt{t}} = c + \frac{a}{\sqrt{(x^2-1)}} \Rightarrow y = cx\sqrt{(x^2-1)} + ax$$

CASE IV: Exact Differential equations:

$$M(x,y)dx + N(x,y)dy = 0$$

The above equation will be exact in nature if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solution will be of the form:

$$P(x) = \frac{(2x^2 - x)}{x^2},\ Q(x) = \frac{1}{x^2}$$

or

$$\underset{\text{[Take only those terms in N which do not contain y]}}{\int Mdx} + \underset{\text{[Taking x in N as constant]}}{\int Ndy} = C$$

Example 15: Solve $(x^2+2xy)dx + (x^2+y^2)dy = 0$

Soln: Here, $M = (x^2+2xy)$ and $N = (x^2+y^2) \Rightarrow \dfrac{\partial M}{\partial y} = 2x$ and $\dfrac{\partial N}{\partial x} = 2x$

Hence, the given equation is exact.

The solution is: $\int (x^2+2xy)dx + \int y^2dy = C \Rightarrow \dfrac{x^3}{3} + x^2y + \dfrac{y^3}{3} = C$

Example 16: Solve $\{y(1+1/x)\cos y\}dx + (x + \log x - x\sin y)dy = 0$

Soln: Here $M = y(1+1/x) + \cos y,\ N = x + \log x - x\sin y$ and $\dfrac{\partial M}{\partial y} = 1 + \dfrac{1}{x} - \sin y = \dfrac{\partial N}{\partial x}$

Solution is $\int (y + y/x + \cos y)dx + \int 0.dy = c \Rightarrow yx + y\log x + x\cos y = c$

List of Exact differentials:

(i) $d\left(\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$ (ii) $d\left(\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}$

(iii) $d\left(\dfrac{y^2}{x}\right) = \dfrac{2xydx - y^2dx}{x^2}$

(iv) $d\left(\dfrac{x^2}{y}\right) = \dfrac{2yxdx - x^2dy}{y^2}$

(v) $d\left(\dfrac{y^2}{x^2}\right) = \dfrac{2x^2ydx - 2xy^2dx}{x^4}$

(vi) $d\left(\dfrac{x^2}{y^2}\right) = \dfrac{2y^2xdx - 2yx^2dy}{y^4}$

(vii) $d\left[\log(xy)\right] = \dfrac{xdy + ydx}{xy}$

(viii) $d(xy) = xdy + ydx$

(ix) $d\left[\tan^{-1}\dfrac{y}{x}\right] = \dfrac{xdy - ydx}{x^2+y^2}$

(x) $d\left[\tan^{-1}\dfrac{x}{y}\right] = \dfrac{ydx - xdy}{x^2+y^2}$

(xi) $d\left[\log\left(\dfrac{x}{y}\right)\right] = \dfrac{xdy - ydx}{xy}$

(xii) $d\left[\log\left(\dfrac{x}{y}\right)\right] = \dfrac{ydx - xdy}{xy}$

(xiii) $d\left[\dfrac{1}{2}\log(x^2+y^2)\right] = \dfrac{xdx + ydy}{x^2+y^2}$

(xiv) $d\left[-\dfrac{1}{xy}\right] = \dfrac{xdy + ydx}{x^2y^2}$

Example 17: Solve $ydx - xdy + (1+x^2)dx + x^2\sin ydy = 0$

Soln: Dividing each term of the given equation by x^2, we get

$$\frac{ydx - xdy}{x^2} + \frac{1+x^2}{x^2}dx + \sin ydy = 0 \Rightarrow -\frac{xdy - ydx}{x^2} + \left(\frac{1}{x^2}+1\right)dx + \sin ydy = 0$$

$$\Rightarrow -d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x^2}\right)dx + \sin ydy = 0$$

Integrating, $-\left(\dfrac{y}{x}\right) + x - \left(\dfrac{1}{x}\right) - \cos y = c \Rightarrow -y + x^2 - 1 - x\cos y = cx$

Orthogonal Trajectories:

Two family of curves are such that every curve of either family cuts each curve of the other family at right angles, then they are called orthogonal trajectories of each other.

Example: (i) The path of a electric field is perpendicular to equipotential surface.

(ii) The line of heat flow is perpendicular to isothermal curves.

Example 18: Find the orthogonal trajectories of the family of curves xy = c

Soln: Given: xy = c $\Rightarrow y + x\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ we get, $-\dfrac{dx}{dy} = -\dfrac{y}{x} \Rightarrow ydy = xdx \Rightarrow \dfrac{y^2}{2} = \dfrac{x^2}{2} + c \Rightarrow y^2 - x^2 = 2c$

Example 19: If y(x) satisfies the differential equation $\frac{dy}{dx}+2y=2+e^{-x^2}$ with y(0) = 0, then $\underset{x\to\infty}{Lt}\, y(x)$ is equal to

(a) 0 (b) 1 (c) 2 (d) -1

Soln: Given differential equation is a first order linear differential equation.

I.F. $= e^{\int 2dx} = e^{2x}$;

Therefore the solution is

$$y.e^{2x}=\int\left(2e^{2x}+e^{2x-x^2}\right)dx \Rightarrow y.e^{2x}=e^{2x}+\int e^{2x-x^2}dx \Rightarrow y=1+e^{-2x}\int e^{2x-x^2}dx \Rightarrow \underset{x\to\infty}{Lt}\, y(x)=1$$

3.3 Linear Second Order Differential Equations

CASE I: Homogeneous Differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+QY=0 \qquad (P, Q, \text{ are constant coefficients})$$

Say, $y = ce^{mx}$ be the trial solution

$$\Rightarrow ce^{mx}(m^2+Pm+Q)=0 \quad \Rightarrow (m^2+Pm+Q)=0$$

The above equation is known as auxilliary equation corresponding to the differential equation.

Suppose m_1, m_2 are two solutions of auxillary equation. Depending on m_1, m_2, there will be varoius forms of solution possible. Now,

(i) If m_1, m_2 are real and unequal, then solution will be of the form:

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

(ii) If m_1, m_2 are real and equal ($m_1 = m_2 = m$), then solution will be of the form:

$$y=(c_1+c_2x)e^{mx}$$

(iii) If m_1, m_2 are complex numbers, then solution will be of the form:

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

(iv) If m_1, m_2 are complex conjugate of each other i.e. $m_1, m_2 = \alpha \pm i\beta$, then solution will be of the form:

$$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

Example 20: Solve $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=0$

Soln: The given differential equation is $\left(D^2-7D+12\right)y=0$ [where $D\equiv\frac{d}{dx}$]

The auxilary equation is $m^2-7m+12=0 \quad \Rightarrow (m-3)(m-4)=0 \Rightarrow m=3, 4$

The complete solution is $y(x)=C_1e^{3x}+C_2e^{4x}$, where C_1 & C_2 are constants.

Example 21: Solve $\frac{d^3y}{dx^3}-8y=0$

Soln: The auxilary equation is $m^3-8=0 \Rightarrow (m-2)(m^2+2m+4)=0 \Rightarrow m=2,-1+i\sqrt{3},-1-i\sqrt{3}$

The complete solution is $y(x)=C_1e^{2x}+e^{-x}\left[C_2\cos\sqrt{3}\,x+C_3\sin\sqrt{3}\,x\right]$

Example 22: Consider the differential equation.

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0; \qquad at\ t=0, x=1\ \&\ \frac{dx}{dt}=0$$

Find the value of x at $t=1$.

Soln: The auxilary equation is $m^2+2m+1=0 \Rightarrow m=-1,-1$

Therefore, the solution is $x(t)=C_1e^{-t}+C_2te^{-t} \Rightarrow \frac{dx}{dt}=-C_1e^{-t}+C_2e^{-t}-C_2t\,e^{-t}$

At $t=0, x=1 \Rightarrow C_1=1$

At $t=0$, $\frac{dx}{dt}=0 \Rightarrow 0=-C_1+C_2 \quad \Rightarrow C_1=C_2=1 \Rightarrow x(t)=e^{-t}+t\,e^{-t}$

At t = 1, the value of x will be $e^{-1}+e^{-1}=\frac{2}{e}$

CASE II: Non-Homogeneous Differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R(x)$$

The above equation can be written as

$$\left(D^2+P.D+Q\right)y=R(x) \Rightarrow f(D)y=R(x) \quad \left[\text{where } D\equiv\frac{d}{dx}\right]$$

The complete solution of the above differential equation consists of two parts:

(i) Complementary function (C.F.): same as the solution of the equivalent homogenous differential equation,

(ii) Particular integral (P.I): depends on the form of $R(x)$

Rules to get Particular Integral:

(1) $R(x)=e^{ax}$, then particular integral $=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$

(i) If $f(a)=0$, then particular integral $=x\frac{1}{f'(a)}e^{ax}$

(ii) If $f'(a)=0$ then particular integral $=x^2\frac{1}{f''(a)}e^{ax}$

(2) $R(x)=$ polynomial of x ; then particular integral $=\frac{1}{f(D)}x^n$ [Expand $\frac{1}{f(D)}$ in series]

(3) $R(x)=\cos ax$ *or* $\sin ax$; then particular integral $=\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax$;

(i) if $f(-a^2)=0$, then particular integral $=\frac{x}{f'(-a^2)}\sin ax$

(4) $R(x)=e^{ax}\phi(x)$; then particular integral $=\frac{1}{f(D)}e^{ax}\cdot\phi(x)=e^{ax}\frac{1}{f(D+a)}\phi(x)$

Example 23: Solve $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{5x}$

Soln: Auxillary equation: $m^2-3m+2=0 \Rightarrow (m-1)(m-2)=0 \Rightarrow m=1,2 \Rightarrow \text{C.F.}=C_1e^x+C_2e^{2x}$

and particular integral (P.I) $=\frac{e^{5x}}{D^2-3D+2}=\frac{e^{5x}}{5^2-3.5+2}=\frac{e^{5x}}{12}$

Therefore, Complete solution is $y=C.F.+P.I.=C_1e^x+C_2e^{2x}+\frac{e^{5x}}{12}$

Example 24: $\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=\sin 2x$

Soln: Auxillary equation is $m^2-m-2=0 \Rightarrow m=-1,2 \Rightarrow C.F.=C_1e^{-x}+C_2e^{2x}$

$$P.I.=\frac{\sin 2x}{D^2-D-2}=\frac{\sin 2x}{-(2)^2-D-2}=\frac{-\sin 2x}{D+6}$$

$$=-\frac{(D-6)\sin 2x}{D^2-36}=-\frac{(D-6)\sin 2x}{-(2)^2-36}=\frac{1}{40}(D-6)\sin 2x=\frac{2\cos 2x-6\sin 2x}{40}$$

Therefore, Complete solution is $y=C.F.+P.I.=C_1e^{-x}+C_2e^{2x}+\frac{1}{20}(\cos 2x-3\sin 2x)$

Example 25: Solve $(D^2-2D+1)y=x^2e^{3x}$

Soln: Auxillary equation is $m^2-2m+1=0 \Rightarrow m=1,1 \Rightarrow C.F.=C_1e^x+C_2xe^x=(C_1+C_2x)e^x$

$$P.I.=\frac{1(x^2e^{3x})}{D^2-2D+1}=\frac{1}{(D-1)^2}e^{3x}x^2=e^{3x}\frac{1\ x^2}{\{(D+3)-1\}^2}=e^{3x}.\frac{1\ x^2}{(D+2)^2}$$

$$=\frac{1}{4}e^{3x}\left(1+\frac{D}{2}\right)^{-2}x^2=\frac{1}{4}e^{3x}\left[1-D+\frac{3}{4}D^2+....\right]x^2 \quad\Rightarrow\quad P.I.=\frac{e^{3x}}{4}\left[x^2-2x+\frac{3}{2}\right].$$

Hence complete solution is y = C.F. + P.I. $\Rightarrow y=(C_1+C_2x)e^x+\frac{e^{3x}}{4}\left[x^2-2x+\frac{3}{2}\right]$

Example 26: Solve $(D^3-D^2-6D)y=x^2+1$

Soln: Auxiliary equation: $m(m^2-m-6)=0 \Rightarrow$ m = 0,3,-2 $\Rightarrow C.F.=c_1+c_2e^{3x}+c_3e^{-2x}$

$$P.I.=\frac{1}{D^3-D^2-6D}(x^2+1)=\frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}(x^2+1)=-\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$=-\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-......\right](x^2+1)$$

$$=-\frac{1}{6D}\left[x^2+1-\frac{1}{6}D(x^2-1)+\frac{7}{36}D^2(x^2+1)+.....\right]=-\frac{1}{6D}\left[x^2+1-\frac{1}{6}(2x+0)+\frac{7}{36}(2+0)+...\right]$$

$$=-\frac{1}{6D}\left[x^2-\frac{1}{3}x+\frac{25}{18}\right]=-\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

Hence, the required solution is $y=C.F.+P.I=y=c_1+c_2e^{3x}+c_3e^{-2x}-\left(\frac{1}{18}\right)\left(x^3-\frac{x^2}{2}+\frac{25x}{6}\right)$

Example 27: Solve $(D^3+8)y=x^4+2x+1$

Soln: The auxialiary equation is $(m+2)(m^2-2m+4)=0 \Rightarrow m=-2,1\pm i\sqrt{3}$

$\Rightarrow C.F.=c_1e^{-2x}+e^x\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)$

$$P.I.=\frac{1}{D^3+8}(x^4+2x+1)=\frac{1}{8\left(1+\frac{1}{8}D^3\right)}(x^4+2x+1)$$

$$=\frac{1}{8}\left(1+\frac{1}{8}D^3\right)^{-1}(x^4+2x+1)=\frac{1}{8}\left(1-\frac{1}{8}D^3+....\right)(x^4+2x+1)$$

$$=\frac{1}{8}\left[(x^4+2x+1)-\frac{1}{8}D^3(x^4+2x+1)\right]\quad=\frac{1}{8}\left[(x^4+2x+1)-3x\right]=\frac{1}{8}(x^4-x+1)$$

$$\Rightarrow y=c_1e^{-2x}+e^x\left(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right)+\left(\frac{1}{8}\right)(x^4-x+1)$$

Example 28: Solve $(D^2-1)y=\cosh x\cos x=\frac{1}{2}(e^x+e^{-x})\cos x$

Soln: Auxialiary equation is $m^2-1=0 \Rightarrow$ m = -1,1 $\Rightarrow$ C.F. $=c_1e^x+c_2e^{-x}$

$$\text{P.I.}=\frac{1}{2}\frac{1}{D^2-1}e^x\cos x+\frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x=\frac{1}{2}e^x\frac{1}{(D+1)^2-1}\cos x+\frac{1}{2}e^{-x}\frac{1}{(D-1)^2-1}\cos x$$

$$=\frac{1}{2}e^x\frac{1}{D^2+2D}\cos x+\frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x=\frac{1}{2}e^x\frac{1}{-1+2D}\cos x+\frac{e^{-x}}{2}\frac{1}{-1-2D}\cos x$$

$$=\frac{e^x}{2}(2D+1).\frac{1}{(2D+1)(2D-1)}\cos x-\frac{e^{-x}}{2}(2D-1)\frac{1}{(2D-1)(2D+1)}\cos x$$

$$=\frac{e^{x}}{2}(2D+1)\frac{1}{(4D^{2}-1)}\cos x-\frac{e^{-x}}{10}(2D-1)\frac{1}{4D^{2}-1}\cos x$$

$$=\frac{e^{x}}{2}(2D+1)\frac{1}{-4-1}\cos x-\frac{e^{-x}}{2}(2D-1)\frac{1}{-4-1}\cos x$$

$$=-\frac{e^{x}}{10}(2D+1)\cos x+\frac{e^{-x}}{10}(2D-1)\cos x=-\frac{e^{x}}{10}(-2\sin x+\cos x)+\frac{e^{-x}}{10}(-2\sin x-\cos x)$$

$$\therefore \quad y=c_1e^{x}+c_2e^{-x}-\left(\frac{1}{10}\right)e^{x}(-2\sin x+\cos x)+\left(\frac{1}{10}\right)e^{-x}(-2\sin x-\cos x)$$

$$\Rightarrow y=c_1e^{x}+c_2e^{-x}+\frac{2}{5}\sin x\left(\frac{e^{x}-e^{-x}}{2}\right)-\frac{1}{5}\cos x\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$\Rightarrow y=c_1e^{x}+c_2e^{-x}+\left(\frac{2}{5}\right)\sin x\sinh x-\left(\frac{1}{5}\right)\cos x\cos hx$$

Example 29: Solve $(D^{2}+2D+1)y=x\cos x$

Soln: Auxiliary equation is $(m+1)^{2}=0 \Rightarrow m=-1,-1$

$$P.I.=\frac{1}{(D+1)^{2}}x\cos x=\text{real part of }\frac{1}{(D+1)^{2}}xe^{ix}$$

$$\text{Now, } \frac{1}{(D+1)^{2}}xe^{ix}=e^{ix}\frac{1}{(D+i+1)^{2}}x=\frac{e^{iax}}{(1+i)^{2}}\frac{1}{\left[1+\frac{D}{1+i}\right]^{2}}.x$$

$$=\frac{e^{ix}}{2i}\left[1+\frac{D}{1+i}\right]^{-2}x=\frac{e^{ix}}{2i}\left[1-\frac{2D}{1+i}+.....\right]x=-\frac{1}{2}ie^{ix}\left[1-\frac{1-i}{2}.2\right]$$

$$=-\frac{1}{2}ie^{ix}[(x-1)+i]=\frac{1}{2}e^{ix}[(x-1)i-1]=-\frac{1}{2}(\cos x+i\sin x)[(x-1)i-1]$$

$$\text{P.I.}=-\frac{1}{2}[-\cos x-(x-1)\sin x]$$

$$\therefore \quad y=(c_1+c_2x)e^{-x}+\left(\frac{1}{2}\right)[\cos x+(x-1)\sin x]$$

Wronskian of the differential equation:

$$W(y_1,y_2,x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

(i) if $W(y_1,y_2,x)=0$, $y_1(x), y_2(x)$ are linearly dependent.

(ii) if $W(y_1,y_2,x)\neq 0$, $y_1(x), y_2(x)$ are linearly independent.

Note: (i) If y_1 and y_2 are two linearly independent solution of a second order linear homogeneous differential equation then their linear combination will also be a solution of that differential equation.

(ii) The value of the Wronskian at different values of x can be related by the following relation:

$$W(x=x_1)=W(x=x_0)e^{-\int_{x_0}^{x_1}P(x)dx}$$

(iii) A differential equation is said to have dimension 'k' if the differential equation has 'k' number of linearly independent solutions.

Cauchy-Euler differential equation of second order:

$$a_2x^{2}\frac{d^{2}y}{dx^{2}}+a_1x\frac{dy}{dx}+a_0y=Q(x)$$

Put $x=e^{z}\Rightarrow ln\,x=z\Rightarrow \frac{1}{x}=\frac{dz}{dx}$

$$\frac{dy}{dx}=\frac{dy}{dx}\cdot\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}\Rightarrow x\frac{dy}{dx}=\frac{dy}{dz}$$

$$\frac{d^{2}y}{dx^{2}}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{1}{x^{2}}\frac{dy}{dz}+\frac{1}{x}\frac{d^{2}y}{dz^{2}}\left(\frac{dz}{dx}\right)=-\frac{1}{x^{2}}\frac{dy}{dz}+\frac{1}{x^{2}}\frac{d^{2}y}{dz^{2}}\Rightarrow x^{2}\frac{dy}{dx^{2}}=-\frac{dy}{dz}+\frac{d^{2}y}{dz^{2}}$$

Then the given equation becomes

$$a_2\frac{d^{2}y}{dz^{2}}+(a_1-a_2)\frac{dy}{dz}+a_0y=Q(z)$$

The above equation is nothing but a linear second order non-homogeneous differential equation

Example 30: $x^{2}\frac{d^{2}y}{dx^{2}}-2x\frac{dy}{dx}-4y=x^{4}$

Soln: $x=e^{z}\Rightarrow \log x=z\Rightarrow \frac{1}{x}=\frac{dz}{dx}$

Putting $x\frac{dy}{dx}=\frac{dy}{dz}$ and $x^{2}\frac{d^{2}y}{dx^{2}}=\frac{d^{2}y}{dz^{2}}-\frac{dy}{dz}$, we get $\Rightarrow \frac{d^{2}y}{dz^{2}}-3\frac{dy}{dz}-4y=e^{4z}$

Auxillary equation: $m^{2}-3m-4=0\Rightarrow m=4,-1\Rightarrow \text{C.F.}=c_1e^{4z}+c_2e^{-z}$

$$\text{Particular integral}=\frac{1}{D^{2}-3D-4}e^{4z}=z\cdot\frac{1}{2D-3}e^{4z}=\frac{z}{5}e^{4z}$$

Total solution is $c_1e^{4z}+c_2e^{-z}+\frac{z}{5}e^{4z}=c_1x^{4}+c_2\frac{1}{x}+\frac{1}{5}x^{4}\log x$

Example 31: $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-20y=(x+1)^2$

Soln: $x=e^z \Rightarrow \log x = z \Rightarrow \frac{1}{x}=\frac{dz}{dx}$

Putting $x\frac{dy}{dx}=\frac{dy}{dz}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$, we get $\Rightarrow \frac{d^2y}{dz^2}+\frac{dy}{dz}-20y=(e^z+1)^2$

$C.F.=c_1e^{4z}+c_2e^{-5z}=c_1x^4+c_2x^{-5}$

$P.I.=\frac{1}{D^2+D-20}(e^z+1)^2=\frac{1}{D^2+D-20}(e^{2z}+2e^z+1)=-\frac{1}{14}e^{2z}-\frac{1}{9}e^z-\frac{1}{20}$

Total solution $y=c_1x^4+c_2x^{-5}-\frac{1}{14}x^2-\frac{1}{9}x-\frac{1}{20}$

Method of Variation of parameters:

Consider the following second order non-homogeneous linear differential equation:

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R(x)$$

Suppose $y_1=e^{m_1x}$ and $y_2=e^{m_2x}$ are two linearly independent solutions of the given differential equation, then

Complementary function = C.F. = Ay_1+By_2

Particular integral = P.I. = uy_1+vy_2

where $u=\int-\frac{y_2R(x)}{y_1y_2'-y_2y_1'}dx$ and $v=\int\frac{y_1R(x)}{y_1y_2'-y_2y_1'}dx$

Example 32: Solve: $\frac{d^2y}{dx^2}+y=\tan x$

Soln: Auxilliary equation: $m^2+1=0 \Rightarrow m=\pm i \Rightarrow C.F.=A\cos x+B\sin x$

In this case, $y_1=\cos x$, $y_2=\sin x$ and $y_1y_2'-y_1'y_2=1$

Particular solution = $u\cos x+v\sin x$

$u=-\int y_2\tan x\,dx=-\int\frac{\sin^2x}{\cos x}dx=-\int\frac{1-\cos^2x}{\cos x}dx$

$=-\int(\sec x-\cos x)dx=-\ln(\sec x+\tan x)+\sin x$

$v=\int y_1\tan x\,dx=\int\sin x\,dx=-\cos x$

Particular solution = $-\cos x.\ln(\sec x+\tan x)$

Total Solution is $y=A\cos x+B\sin x-\cos x.\ln(\sec x+\tan x)$

3.4 Singular Points and Ordinary Points

Consider the differential equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

Ordinary point:

If $P(x)$ and $Q(x)$ remain finite at $x=x_0$, then $x=x_0$ is ordinary point of the differential equation.

Singular point:

If either $P(x)$ or $Q(x)$ or both are infinite at $x=x_0$, then $x=x_0$ is singular point of the differential equation. There are two types of singular points i.e.

(i) Regular singular point: If $P(x)$, $Q(x)$ diverges but $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ remain finite at $x=x_0$, then $x=x_0$ is called a regular or non-essential singular points of the differential equation.

(ii) Irregular singular point: If $P(x)$, $Q(x)$ diverges but either $(x-x_0)P(x)$ or $(x-x_0)^2Q(x)$ or both are infinite at $x=x_0$, then $x=x_0$ is called a irregular or essential singular point of the differential equation.

Example:

(i) $(1+x^2)y''+xy'-y=0 \Rightarrow y''+\frac{x}{1+x^2}y'-\frac{y}{1+x^2}=0$

$$P(x)=\frac{x}{1+x^2},\ Q(x)=\frac{-1}{1+x^2}$$

at $x=0$ both P and Q are finite, therefore $x=0$ is an ordinary point of the above differential equation.

(ii) $x^2y''+(2x^2-x)y'+y=0 \Rightarrow y''+\frac{(2x^2-x)}{x^2}y'+\frac{1}{x^2}y=0$

$$P(x)=\frac{(2x^2-x)}{x^2},Q(x)=\frac{1}{x^2}$$

at $x=0$ both P and Q are infinite, then, $x=0$ is an sigular point of the above differential equation.

Example 33: Find the singular point and the corresponding nature of singularity of the following differential equation $x(x-2)^2y''+2(x-2)y'+(x+3)y=0$

Soln: $y''+\frac{2}{x(x-2)}y'+\frac{(x+3)}{x(x-2)^2}y=0 \quad \Rightarrow P(x)=\frac{2}{x(x-2)},\ Q(x)=\frac{x+3}{x(x-2)^2}$

$P(x)$ and $Q(x)$ are infinite at points $x=0, x=2$.

$(x-2)P(x)$ & $(x-2)^2Q(x)$ are finite at point $x=2$.

$xP(x)$ and $x^2Q(x)$ are finite at point $x=0$

Therefore, $x=0$ and $x=2$ are regular singular points of the given differential equation.

3.5 Power series solution (About Ordinary Points)

Example 34: Find the power series solution of $\left(1-x^2\right)y''-2xy'+2y=0$ about $x=0$

Soln: Since $x=0$ is an ordinary point of the given differential equation, the solution can be written as

$$y=\sum_{k=0}^{\infty}a_k x^k \Rightarrow \frac{dy}{dx}=\sum_{k=0}^{\infty}k\,a_k x^{k-1} \Rightarrow \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}a_k k(k-1)x^{k-2}$$

Substituting these values in the given equation we get.

$$\left(1-x^2\right)\sum_k a_k k(k-1)x^{k-2}-2x\sum_k a_k(k)x^{k-1}+2\sum_k a_k x^k=0$$

$$\Rightarrow \quad \sum_{k=2}a_k k(k-1)x^{k-2}-\sum\left(k^2+k-2\right)a_k x^k=0$$

Equating the coefficient of x^k to be zero, we get

$$(k+2)(k+1)\;a_{k+2}-\left(k^2+k-2\right)a_k=0 \Rightarrow a_{k+2}=\frac{k-1}{(k+1)}a_k$$

For $k=0 \Rightarrow a_2=-a_0;$ $\qquad k=1\Rightarrow a_3=0;$

$k=2 \Rightarrow a_4=\frac{a_2}{3}=\frac{-a_0}{3};$ $\qquad k=3\Rightarrow a_5=\frac{2}{4}a_3=0$

Therefore, solution $y=a_0+a_1x+a_2x^2+\ldots\ldots=a_0\left[1-x^2-\frac{x^4}{3}\ldots..\right]+a_1x$

3.6 Frobenius Method (About Regular Singular point)

Example 35: Solve $x(x-1)y''+(3x-1)y'+y=0$

Soln: Since $x=0$ is a regular singular point, we assume the solution

$$y=\sum_{k=0}^{\infty}a_k x^{m+k} \Rightarrow \frac{dy}{dx}=\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1} \Rightarrow \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^m$$

Putting these in given equation, we get

$$\sum_{k=0}^{\infty}x(x-1)a_k(m+k)(m+k-1)x^{m+k-2}+(3x-1)\sum_k a_k(m+k)x^{m+k-1}+\sum a_k x^{m+k}=0$$

$$\Rightarrow \sum_k a_k\left[(m+k)(m+k+2)+1\right]x^{m+k}-\sum a_k(m+k)^2x^{m+k-1}=0$$

Now the coefficient of lowest power of x i.e. x^{m-1} is $a_0(m+0)^2=0$ (Indicial equation)

$\Rightarrow \quad m=0,0$ as $a_0\neq 0$

Now coefficient of next power of x i.e. x^m is

$$a_1\left[(m+0)(m+2)+1\right]-a_0(m+0)^2=0 \Rightarrow a_1-a_0=0\Rightarrow a_1=a_0$$

Now the coefficient of general power of x i.e. x^{m+k} is:

$$a_k\left[(m+k)(m+k+2)+1\right]-a_{k+1}(m+k+1)^2=0 \Rightarrow a_{k+1}=a_k$$

Solution of the differential equation is $y=a_0x^m\left[1+x+x^2+\ldots\ldots\right]$, but m = 0, 0 gives only one solution instead two.

$$\therefore \quad y=a_0\left[1+x+x^2+\ldots\ldots\right]$$

3.6 Legendre Differential Equation

$$\boxed{\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0} \qquad (1)$$

Note: $x=\pm1$ are the singular points and $x=0$ is the ordinary point of the differential equation

Solution:

Assume, $\quad y=\sum_{r=a}^{\infty}a_r x^{k-r} \Rightarrow \frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k-r)x^{k-r-1}$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}$$

Substituting all these in the Legendre differential equation, we get

$$\left(1-x^2\right)\sum_{r=0}^{\infty}a_r(k-r)(k-r-1)x^{k-r-2}-2x\sum_{r=0}^{\infty}a_r(k-r)x^{k-r-1}+n(n+1)\sum_{r=0}^{\infty}a_r x^{k-r}=0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty}a_r\left[(k-r)(k-r-1)x^{k-r-2}\right]+\sum_{r=0}^{\infty}a_r\left[n(n+1)-(k-r)(k-r-1)-2(k-r)\right]x^{k-r}=0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty}a_r\left[(k-r)(k-r-1)x^{k-r-2}\right]+\sum_{r=0}^{\infty}a_r\left[n(n+1)-(k-r)(k-r+1)\right]x^{k-r}=0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty}a_r\left[(k-r)(k-r-1)x^{k-r-2}\right]+\sum_{r=0}^{\infty}a_r\left[n^2+n-(k-r)^2-(k-r)\right]x^{k-r}=0$$

$$\Rightarrow \quad \sum_{r=0}^{\infty}a_r\left[(k-r)(k-r-1)x^{k-r-2}\right]+\sum_{r=0}^{\infty}a_r\left[(n-k+r)(n+k-r+1)\right]x^{k-r}=0 \qquad (2)$$

Comparing the co-efficients of the highest power of 'x' i.e. of x^k, we get

$$a_0\left[(n-k)(n+k+1)\right]=0 \qquad (3)$$

$$\Rightarrow \quad k=n \text{ or } k=-(n+1)$$

Comparing the co-efficient of next lower power i.e. x^{k-1}, we get,

$$a_1(n-k+1)(n+k)=0$$

According to equation (3), $k=n$ or $k=-(n+1)$

(4)

So, $a_1 = 0$

Comparing the co-efficient of x^{k-r-2}, we get

$$a_r(k-r)(k-r-1)+a_{r+2}(n-k+r+2)(n+k-r-1)=0$$

$$\Rightarrow \quad \boxed{a_{r+2}=\frac{(k-r)(k-r-1)}{(k-n-r-2)(k+n-r-1)}a_0} \qquad (5)$$

Since, $a_1 = 0$ then $a_3 = a_5 = a_7 = ... = 0$

Case-I: k = n

From equation (5), we get

$$a_{r+2}=\frac{(n-r)(n-r-1)}{(-r-2)(2n-r-1)}a_r$$

Putting $r = 0, 2, 4$, we get

$$a_2=-\frac{n(n-1)}{2(2n-1)}a_0$$

$$a_4=-\frac{(n-2)(n-3)}{4(2n-3)}a_2=\frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdot(2n-1)(2n-3)}a_0$$

Therefore, the solution of the equation (1) will be of the form:

$$y=a_0x^n+a_2x^{n-2}+a_4x^{n-4}+........$$

$$\Rightarrow \quad y=a_0\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdot(2n-1)(2n-3)}x^{n-4}........\right] \qquad (6)$$

Solution given by equation (6) is known as Legendre function of first kind. Since, this is an terminating series, it is known as Legendre polynomial of order 'n', i.e. $P_n(x)$

Case II: k = - (n+1)

$$a_{r+2}=\frac{(-n-1-r)(-n-r-2)}{(-2n-r-3)(-r-2)}a_r=\frac{(n+r+1)(n+r+2)}{(2n+r+3)(r+2)}a_r$$

Putting $r = 0, 2, 4$ we get,

$$a_2=\frac{(n+1)(n+2)}{(2n+3)(2)}a_0$$

$$a_4=\frac{(n+3)(n+4)}{(2n+5)(4)}a_2=\frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}a_0$$

Therefore, the solution will be of the form

$$y=a_0x^{-n-1}+a_2x^{-n-3}+a_4^{-n-5}+..........$$

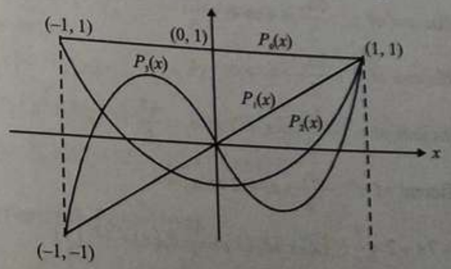
$$=a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-n-5}+.......\right] \qquad (7)$$

solution given by equation (7) is a non-terminating series and known as Legendre function of second kind i.e. $Q_n(x)$

Rodrigue's Formula for Legendre's Polynomial $P_n(x)$:

Legendre polynomial of order 'n' : $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$P_0(x)=1$; $\quad P_1(x)=x$; $\quad P_2(x)=\frac{1}{2}(3x^2-1)$; $\quad P_3(x)=\frac{1}{2}(5x^3-3x)$



Properties of Legendre's polynomial:

(1) $P_n(1)=1$ $\qquad$ (2) $P_n(-1)=(-1)^n$

(3) $P_n(x)$ = even function, if n = even

= odd function if n = odd

(4) $P_n(-x)=(-1)^n P_n(x)$

Generating function of Legendre's polynomial:.

$$\boxed{(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty}z^nP_n(x)}$$

Co-efficient of z^n in the exapansion of the generating function is the Legendre's polynomial of order n.

Orthogonal properties of Legendre's polynomial:

$$\int_{-1}^{+1}P_m(x)P_n(x)\,dx=\frac{2}{2n+1}\delta_{mn}$$

Recurrence relations for Legendre's polynomial:

(i) $nP_n=(2n-1)xP_{n-1}-(n-1)P_{n-2}$ (ii) $nP_n=xP_n'-P_{n-1}'$

(iii) $(2n+1)P_n=P_{n+1}'-P_{n-1}'$ (iv) $(n+1)P_n=P_{n+1}'-xP_n'$

(v) $(1-x^2)P_n'=n(P_{n-1}-xP_n)$ (vi) $(1-x^2)P_n'=(n+1)(xP_n-P_{n+1})$

Example 36: Express $4x^3+6x^2+7x+2$ in terms of Legendre Polynpmials.

Soln: Let, $4x^3+6x^2+7x+2=a_3P_3(x)+a_2P_2(x)+a_1P_1(x)+a_0P_0(x)$

$$=\frac{a_3}{2}(5x^3-3x)+\frac{a_2}{2}(3x^2-1)+a_1x+a_0$$

Comparing the co-efficient of $x^3:\frac{5a_3}{2}=4\Rightarrow a_3=\frac{8}{5}$

Comparing the co-efficient of $x^2:\frac{3a_2}{2}=6\Rightarrow a_2=4$

Comparing the co-efficient of $x^1:-\frac{3a_3}{2}+a_1=7\Rightarrow a_1=\frac{47}{5}$

Comparing the co-efficient of $x^0:-\frac{a_2}{2}+a_0=2\Rightarrow a_0=4$

Therefore, $4x^3+6x^2+7x+2=\frac{8}{5}P_3(x)+4P_2(x)+\frac{47}{5}P_1(x)+4P_0(x)$

Example 37: Show that, $\int_{-1}^{1}x^2P_{n+1}P_{n-1}dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

Soln: Using the relation: $\boxed{(2n+1)xP_n=(n+1)P_{n+1}+nP_{n-1}}$

Putting (n+1) in place of n $\Rightarrow(2n+3)xP_{n+1}=(n+2)P_{n+2}+(n+1)P_n$

Putting (n-1) in place of n $\Rightarrow(2n-1)xP_{n-1}=nP_n+(n-1)P_{n-2}$

$$\int_{-1}^{1}(2n+3)(2n-1)x^2P_{n+1}P_{n-1}dx=n(n+2)\int_{-1}^{1}P_{n+2}P_ndx+n(n+1)\int_{-1}^{1}P_nP_ndx+$$
$$(n-1)(n+2)\int_{-1}^{1}P_{n-2}P_{n+2}dx+\int_{-1}^{1}P_nP_{n-2}dx$$

$$\Rightarrow\int_{-1}^{1}x^2P_{n-1}P_{n+1}dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Example 38: Show that $\sum_n P_n(x)=\frac{1}{\sqrt{2-2x}}$

Soln: $(1-2xz+z^2)^{-1/2}=\sum_n z^nP_n(x)$

Putting $z=1$ in equation, $(1-2x+1)^{-1/2}=\sum_n P_n(x)$ $\Rightarrow\sum_n P_n(x)=\frac{1}{\sqrt{2-2x}}$

Example 39: Prove that $P'_{n+1}(x)+P'_n(x)=P_0+3P_1+5P_2+......+(2n+1)P_n$

Soln: Using the relation $(2n+1)P_n(x)=P'_{n+1}(x)-P'_{n-1}(x)$

putting n = 1 $\Rightarrow 3P_1(x)=P'_2(x)-P'_0(x)$

putting n = 2 $\Rightarrow 5P_2(x)=P'_3(x)-P'_1(x)$

putting n = 3 $\Rightarrow 7P_3(x)=P'_4(x)-P'_2(x)$ and so on.

Adding all the equations we get,

$$3P_1(x)+5P_2(x)+7P_3(x)+......(2n+1)P_n(x)$$
$$=-P'_0(x)-P'_1(x)+P'_n(x)+P'_{n+1}(x)=0-P_0(x)+P'_n(x)+P'_{n+1}(x)$$
$$\Rightarrow P'_{n+1}(x)+P'_n(x)=P_0+3P_1+5P_2+......+(2n+1)P_n$$

3.8 Bessel Differential Equation

$$\boxed{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0} \tag{8}$$

Note: $x=0$ is a regular singular point of the Bessel differential equation.

Solution:

$$y=\sum_{r=0}^{\infty}a_rx^{k+r}\Rightarrow\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}$$

Substituting all these in equation (8), we get

$$x^2\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}+x\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}+(x^2-n^2)\sum_{r=0}^{\infty}a_rx^{k+r}=0$$

$$\Rightarrow\quad\sum_{r=0}^{\infty}a_r\left[(k+r)(k+r-1)+(k+r)-n^2\right]x^{k+r}+\sum_{r=0}^{\infty}a_rx^{k+r+2}=0$$

$$\Rightarrow\quad\sum_{r=0}^{\infty}a_r\left[(k+r)^2-(k+r)+(k+r)-n^2\right]x^{k+r}+\sum_{r=0}^{\infty}a_rx^{k+r+2}=0$$

$$\Rightarrow\quad\sum_{r=0}^{\infty}a_r\left[(k+r)^2-n^2\right]x^{k+r}+\sum_{r=0}^{\infty}a_rx^{k+r+2}=0$$

Comparing the co-efficient of lowest power of x i.e. of x^k we get

$$a_0\left(k^2-n^2\right)=0 \;\Rightarrow k=\pm n \tag{9}$$

Comparing the next higher power of x i.e. x^{k+1} we get

$$a_1\left[(k+1)^2-n^2\right]=0 \;\Rightarrow a_1=0 \tag{10}$$

Comparing the co-efficient of x^{k+r+2}, we get

$$a_{r+2}\left[(k+r+2)^2-n^2\right]+a_r=0$$

$$\Rightarrow \quad \boxed{a_{r+2}=-\frac{a_r}{(k+r+2)^2-n^2}} \tag{11}$$

Since, $a_1=0$, then $a_3=a_5=a_7=0$........

Case 1 : k = n

$$a_{r+2}=-\frac{a_r}{(n+r+2)^2-n^2}$$

Putting $r=0,2,4,\ldots\ldots$ we get,

$$a_2=-\frac{a_0}{(n+2)^2-n^2}=-\frac{a_0}{(2n+2)\cdot 2}=-\frac{a_0}{2^2\cdot(n+1)}$$

$$a_4=-\frac{a_2}{(n+4)^2-n^2}=-\frac{a_2}{(2n+4)\cdot 4}=\frac{a_0}{2^4(n+1)(n+2)\cdot 2!}$$

The solution will be of the form,

$$y=a_0x^n+a_2x^{n+2}+a_4x^{n+4}\ldots\ldots$$

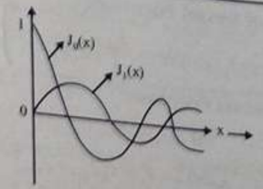
$$\Rightarrow \quad y=a_0\left[x^n-\frac{1}{2^2\cdot(n+1)}x^{n+2}+\frac{1\cdot x^{n+4}}{2^4\cdot 2!(n+1)(n+2)}\ldots\ldots\ldots\right] \tag{12}$$

Solution given by eqution (12) is a non-terminating series and it is known as Bessel function of first kind i.e. $J_n(x)$.

$$\boxed{J_n(x)=y=a_0\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}} \text{ where } a_0=\frac{1}{2^n\lceil n+1}$$

$$n=0 \quad \Rightarrow \quad J_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r} = 1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots..$$

$$n=1 \quad \Rightarrow \quad J_1(x) = \frac{x}{2}-\frac{x^3}{2^24}+\frac{x^5}{2^2\cdot 4^2\cdot 6}-\ldots..$$



Properties of Bessel function of first kind:

(i) $J_n(x)$ are oscillatory function with varying period and decreasing amplitude.

(ii) $J_n(x)$ is an even function when n is even and $J_n(x)$ is an odd function when n is odd.

(iii) $J_0(x)$ has nodes at $x=2.4048, 5.5201$ .... and $J_1(x)$ has nodes at $x=3.8317, 7.015$ ....

(iv) $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

Case 2: k = - n

$$a_{r+2}=-\frac{a_r}{(-n+r+2)^2-n^2}$$

Putting $r=0,2,4,\ldots\ldots\ldots$ we get

$$a_2=-\frac{a_0}{(-n+2)^2-n^2}=-\frac{a_0}{(2n-2)(-2)}=\frac{a_0}{2^2\cdot(n-1)}$$

$$a_4=-\frac{a_2}{(-n+4)^2-n^2}=-\frac{a_2}{(2n-4)(-4)}=\frac{a_0}{2^4\cdot 2!(n-1)(n-2)}$$

The solution will be of the form:

$$y=a_0x^{-n}+a_2x^{-n+2}+a_4x^{-n+4}+\ldots\ldots$$

$$=a_0\left[x^{-n}+\frac{1}{2^n(n-1)}x^{-n+2}+\frac{1}{2^4\cdot 2!(n-1)(n-2)}x^{-n+4}+\ldots..\right] \tag{13}$$

Solution given by equation (13) is a non-terminating series and known as Bessel function of 2nd kind i.e. $J_{-n}(x)$

Generating function of Bessel function:

$$\boxed{e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}=\sum_{n=-\infty}^{\infty}z^nJ_n(x)}$$

The coefficient of z^n in the expansion of the generating function is the Bessel function of order n.

Orthonormality condition of Bessel Fucntion:

$$\boxed{\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\delta_{\alpha\beta}}{2}\left[J_{n+1}(\alpha)\right]^2}$$

Recurrence relations for Bessel Fucntion:

(i) $xJ_n'(x) = nJ_n - xJ_{n+1}$ (ii) $xJ_n' = -nJ_n x J_{n-1}$

(iii) $2J_n' = J_{n-1} - J_{n+1}$ (iv) $2n J_n = x\left[J_{n-1} + J_{n+1}\right]$

(v) $\frac{d}{dx}\left[x^{-n}J_n\right] = -x^{-n}J_{n+1}$ (vi) $\frac{d}{dx}\left[x^{n}J_n\right] = x^{n}J_{n-1}$

Example 40: Show that, $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2\left[\frac{n}{x}J_n^2(x) - \frac{n+1}{x}J_{n+1}^2(x)\right]$

Soln: $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2J_nJ_n' + 2J_{n+1}J_{n+1}' = 2J_n\left[\frac{n}{x}J_n - J_{n+1}\right] + 2J_{n+1}\left[J_n - \frac{n+1}{x}J_{n+1}\right]$

$$= 2\left[\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right]$$

Example 41: Show that $\sqrt{\frac{\pi x}{2}}\,J_{3/2}(x) = \frac{\sin x}{x} - \cos x$

Soln: Using recurrence relation, $2n J_n = x\left[J_{n+1} + J_{n-1}\right]$

Putting $n = ½$, we get, $J_{½} = x\left[J_{3/2} + J_{-1/2}\right]$

$\Rightarrow x J_{3/2} = J_{1/2} - xJ_{-1/2}$ $\Rightarrow \sqrt{\frac{\pi x}{2}}J_{3/2} = \frac{\sin x}{x} - \cos x$ (using the expressions of $J_{½}$ & $J_{-½}$)

3.9 Hermite Differentiation Equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \tag{14}$$

Note: $x = 0$ is an ordinary point of the Hermite differential equation

Solution:

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \Rightarrow \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r)x^{k+r-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1)x^{k+r-2}$$

Substituting all these in equation (14) we have,

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-2} - 2(k+r)x^{k+r} + 2nx^{k+r}\right] = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-2} - 2(k+r-n)x^{k+r}\right] = 0 \tag{15}$$

Equating the coefficient of lowest power of x i.e. of x^{k-2} to be equal to zero i.e.

$$a_0 k(k-1) = 0 \quad \Rightarrow k = 0,1$$

Equating to zero the coefficient of x^{k-1} in equation (15), we have

$$a_1(k+1)k = 0 \Rightarrow a_1 = 0 \text{ or } k = 0 \text{ or both are zero}$$

Equating to zero the coefficient of general term i.e. x^{k+r} in equation (15), we have

$$a_{r+2}(k+r+2)(k+r+1) - 2a_r(k+r-n) = 0$$

$$a_{r+2} = \frac{2(k+r-n)}{(k+r+2)(k+r+1)}\cdot a_r \tag{16}$$

Case I : k = 0

$$a_{r+2} = \frac{2r-2n}{(r+2)(r+1)}a_r \tag{17}$$

Putting $r = 0, 2, 4$ ..... in equation (17), we have

$$a_2 = \frac{-2n}{2\cdot 1}a_0 = -\frac{2n}{2!}a_0$$

$$a_4 = \frac{4-2\lambda}{4.3}a_2 = -\frac{(4-2n).2n}{4.3.2!}a_0 = \frac{(-2)^2(-2+n)n}{4!}a_0$$

and so on.

Therefore, $a_{2m} = \frac{(-2)^m n(n-2)....(n-2m+2)}{(2m)!}a_0$

Again putting $r = 1, 3, 5$ ...... in equation (17), we have

$$a_3 = \frac{2-2n}{3.2}a_1 = -\frac{2(n-1)}{3!}a_1$$

$$a_5 = \frac{6-2n}{5.4}a_3 = -\frac{2(6-2n)(n-1)}{5.4.3!}a_1 = (-2)^2\frac{(n-1)(n-3)}{5!}a_1$$

and so on.

Therefore, $a_{2m+1} = \frac{(-2)^m(n-1)(n-3).....(n-2m+1)}{(2m+1)!}a_1$

Now, if $a_1 \neq 0$, then we have,

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

$$= a_0\left[1 - \frac{2n}{2!}x^2 + \frac{2^2 n(n-2)}{4!}x^4 + ... + \frac{(-2)^m n(n-2)...(n-2m+2)}{(2m)!} + ...\right]$$

$$+a_1\left[x-\frac{2(n-1)}{3!}x^3+\frac{2^2(n-1)(n-3)}{5!}x^5+.....+\frac{(-2)^m(n-1)(n-3)....(n-2m+1)}{(2m+1)!}x^{2m+1}+....\right] \tag{18}$$

and if $a_1=0$, then we have

$$y=a_0\left[1-\frac{2n}{2!}x^2+\frac{2^2n(n-2)}{4!}x^4+......+\frac{(-2)^m n(n-2)...(n-2m+2)}{(2m)!}x^{2m}+....\right] \tag{19}$$

$$=y_1(say)$$

Case-II: k = 1

$$a_{r+2}=\frac{2(r+1)-2n}{(r+3)(r+2)}a_r \tag{20}$$

Putting $r=1,3.......$ in equation (20), we have

$$a_3=a_5=.....=0 \text{ (each)}$$

Putting $r=0,2,4........$ in equation (20), we have

$$a_2=\frac{2+2n}{3.2}a_0=-\frac{2(n-1)}{3!}a_0$$

$$a_4=\frac{6-2n}{5.4}a_2=\frac{2(n-1)(n-3)}{5!}a_0$$

and so on.

Therefore, $$a_{2m}=\frac{(-2)^m(n-1)(n-3)...(n-2m+1)}{(2m+1)!}a_0$$

Therefore, the solution will be of the form:

$$y=\sum_{r=0}^{\infty}a_r x^{r+1}=a_0x+a_2x^3+a_4x^5+...+a_{2m}x^{2m+1}+......$$

$$=a_0\left\{x-\frac{2(n-1)}{3!}x^2+\frac{2^2(n-1)(n-3)}{5!}x^5-....+\frac{(-2)^m(n-1)(n-3)....(n-2m+1)}{(2m-1)!}x^{2m+1}+......\right\} \tag{21}$$

$$=y_2(say)$$

Hence, the general solution of Hermite's differential equation is

$$y=Ay_1+By_2$$

where A and B are arbitrary constants.

Hermite's Polynomials:

When λ is an even integer, equation (19) gives an even polynomial of degree n.

Let $\lambda=n$, n is an even integer.

and $$a_0=(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!}$$

Therefore, coefficient of x^n in equation (19) is

$$(-1)^{n/2}\frac{n!}{\left(\frac{n}{2}\right)!}\cdot\frac{(-2)^{n/2}n(n-2).....(n-n+2)}{n!}=2^n\cdot\frac{n/2(n/2-1)...1}{(n/2)!}=2^n$$

Similarly coefficient of x^{n-2}

$$=(-1)^{n/2}\frac{n!}{(n/2)!}\frac{(-2)^{(n-2)/2}n(n-2).....(n-n+2+2)}{(n-2)!}$$

$$=-\frac{2^{n-2}n(n-1)\frac{n}{2}\left(\frac{n}{2}-1\right)....2}{\left(\frac{n}{2}\right)!}=-\frac{n(n-1)}{1!}2^{n-2}$$

and so on.

Therefore,

$$y_n=(2x)^2-\frac{n(n-1)}{1!}(2x)^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}....+(-1)^{n/2}\frac{n!}{(n/2)!}$$

This value of y is known as the Hermite's polynomial of degree n and is written as

$$H_n(x)=(2x)^n-\frac{n(n-1)}{1!}(2x)^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}+....+(-1)^{n/2}\frac{n!}{(n/2)!}$$

$$\Rightarrow H_n(x)=\sum_{r=0}^{(n/2)}(-1)^r\frac{n!}{r!(n-2r)!}\cdot(2x)^{n-2r}$$

where, $$\left(\frac{n}{2}\right)=\begin{cases}\frac{n}{2} & \text{is } n \text{ is even}\\ \frac{1}{2}(n-1) & \text{is } n \text{ is odd}\end{cases}$$

Rodrigue's formula for Hermite polynomial:

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$H_0(x)=1 \qquad H_3(x)=8x^3-12x$$

$$H_1(x)=2x \qquad H_4(x)=16x^4-48x^2+12$$

$$H_2(x)=4x^2-2$$

Note: $H_n(x)$ will be even function if n is even and will be odd function if n is odd.

Generating function for Hermite polynomial:

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

Coefficient of z^n in the expansion of e^{2zx-z^2} is $\frac{H_n(x)}{n!}$

Orthogonal property of Hermite polynomial:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{mn}$$

Recurrence relation for Hermite polynomial:

(i) $H_n'(x) = 2n\, H_{n-1}(x)$ (ii) $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

(iii) $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$ (iv) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$

(v) $H_n(-x) = (-1)^n H_n(x)$

3.10 Lagauerre Differential Equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0 \qquad (22)$$

Note: $x = 0$ is a regular singular point of the Laguerre differential equation.

Solution:

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \Rightarrow \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

Substituting all these in equation (22), we have

$$\sum_{r=0}^{\infty} a_r \left[(k+r)(k+r-1)x^{k+r-1} + (1-x)(k+r)x^{k+r-1} + nx^{k+r}\right] = 0$$

$$\sum_{r=0}^{\infty} a_r \left[(k+r)^2 x^{k+r-1} - (k+r-n)x^{k+r}\right] = 0 \qquad (23)$$

Equating the coefficient of lowest power of x i.e. of x^{k-1} to be equal to zero, we have

$$a_0 k^2 = 0 \Rightarrow k = 0, 0 \ (a_0 \neq 0)$$

Equating the coefficient of general term, i.e. of x^{k+r} to be equal to zero, we have

$$a_{r+1}(k+r+1)^2 - a_r(k+r-n) = 0$$

$$\Rightarrow a_{k+1} = \frac{(k+r-n)}{(k+r+1)^2} a_r$$

for $k = 0$, $\quad a_{r+1} = \frac{r-n}{(r+1)^2} a_r \qquad (24)$

Putting $r = 0, 1, 2, ...$ in equation (24), we have

$$a_1 = \frac{-n}{1} a_0 = (-1)na_0$$

$$a_2 = \frac{1-n}{2^2} a_1 = (-1)^2 \frac{n(n-1)}{(2!)^2} a_0$$

$$a_3 = \frac{2-n}{3^2} \cdot a_2 = (-1)^3 \frac{n(n-1)(n-2)}{(3!)^2} a_0, \text{ etc.}$$

$$a_r = (-1)^r \frac{n(n-1)....(n-r+1)}{(r!)^2} a_0$$

Therefore, the solution will be of the form

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + ...a_r x^r + ...$$

$$= a_0 \left[1 - nx + \frac{n(n-1)}{(2!)^2} x^2 + ... + (-1)^r \frac{n(n-1).....(n-r+1)}{(r!)^2} + ...\right]$$

$$y = a_0 \left[1 - \frac{n}{1^2} x + \frac{n(n-1)}{(2!)^2} x^2 + ... + (-1)^r \frac{n(n-1)...(n-r+1)}{(r!)^2} x^r + ...\right]$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n(n-1)...(n-r+1)}{(r!)^2} x^r$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r$$

Rodrigue's formula for Lagauerre polynomial:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

$L_0(x) = 1 \quad L_1(x) = 1 - x \quad L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$

$L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$

Note: Laguerre polynomials are neither even nor odd function of x.

Generating function for Laguerre polynomial:

$$\frac{e^{-xz/(1-z)}}{(1-z)}=\sum_{n=0}^{\infty} z^n L_n(x)$$

Orthogonal property of Laguerre polynomial:

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,dx=\delta_{mn}$$

CSIR-NET/JRF Previous Year Questions with Solutions

1. Let $p_n(x)$ (where $\mathrm{n}=0,1,2,.....$) be ploynomial of order 'n' wirh real co-efficients defined in the interval $2\le n\le 4$. If $\int_2^4 p_m(x)p_n(x)dx=\delta_{mn}$, then **[CSIR June -2011]**

(a) $p_0(x)=\frac{1}{\sqrt{2}}$ and $\mathrm{p}_1(x)=\sqrt{\frac{3}{2}}(-3-x)$ (b) $p_0(x)=\frac{1}{\sqrt{2}}$ and $\mathrm{p}_1(x)=\sqrt{3}(3+x)$

(c) $p_0(x)=\frac{1}{2}$ and $\mathrm{p}_1(x)=\sqrt{\frac{3}{2}}(3-x)$ (d) $p_0(x)=\frac{1}{\sqrt{2}}$ and $\mathrm{p}_1(x)=\sqrt{\frac{3}{2}}(3-x)$

Soln. Evaluate $\int_2^4 \mathrm{p}_0(\mathrm{x})\mathrm{p}_1(\mathrm{x})\mathrm{dx}$ (Orhthogonality condition)

with $\mathrm{p}_0(\mathrm{x})=\frac{1}{\sqrt{2}}$, $\mathrm{p}_1(\mathrm{x})=\frac{\sqrt{3}}{\sqrt{2}}(3-\mathrm{x})$ [Option (d)]

$$=\frac{\sqrt{3}}{4}\int_2^4(3-\mathrm{x})\mathrm{dx} \Rightarrow \frac{\sqrt{3}}{4}\left|3\mathrm{x}-\frac{\mathrm{x}^2}{2}\right|_2^4$$

$$=\frac{\sqrt{3}}{4}\left[\left((12-8)-(6-2)\right)\right]=0$$

Option (c) is eliminated because $\mathrm{p}_0(x)$ is not normalized.

Normalization of $\mathrm{p}_0(x)$: $\mathrm{N}^2\int_2^4|\mathrm{p}_0(\mathrm{x})|^2\,\mathrm{dx}=1 \Rightarrow \mathrm{N}^2|\mathrm{x}|_2^4=1 \Rightarrow 2\mathrm{N}^2=1 \Rightarrow \mathrm{N}=\frac{1}{\sqrt{2}}$

$\therefore$ Normalized $\mathrm{p}_0(\mathrm{x})=\frac{1}{\sqrt{2}}$

Correct answer is (d)

2. The genarating function $F(x,t)=\sum_{n=0}^{\infty}P_n(x)t^n$ for Legendre polynomials $P_n(x)$ is $F(x,t)=\left(1-2xt+t^2\right)^{-1/2}$. The value of $P_3(-1)$ is **[CSIR Dec -2011]**

(a) 5/2 (b) 3/2 (c) 1 (d) -1

Soln. $F(x,t)=\left(1-2xt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)t^n$

Putting $x=-1$ in the above equation, we get

$$\left(1+2t+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(-1)t^n \Rightarrow (1+t)^{-1}=1-t+t^2-t^3+t^4-.....=\sum_{n=0}^{\infty}P_n(-1)t^n$$

Comparing the coefficient t^3 on the both sides, we get $P_3(-1)=-1$

Correct answer is (d)

3. Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of the differential equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+f(t)x=0$. Let, $w(t)=x_1(t)\frac{dx_2(t)}{dt}-x_2(t)\frac{dx_1(t)}{dt}$. If $w(0)=1$, then $w(1)$ is given by

(a) 1 (b) e^2 (c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$ **[CSIR Dec -2011]**

Soln. Wronskian of a second order differential equation is defined as

$$w(t)=x_1(t)\frac{dx_2(t)}{dt}-x_2(t)\frac{dx_1(t)}{dt}=w(t)=x_1x'_2-x_2x'_1$$

Since, $w(t)=w(t_0)e^{-\int_{t_0}^{t}P(t)dt}$

Therefore, $w(t=1)=w(t=0)e^{-\int_0^1 2.dt}=\frac{1}{e^2}$

Correct answer is (d)

4. Let y(x) be a continuous real function in the range of 0 to 2π, satisfying the inhomogeneous differential equation: $\sin x\frac{d^2y}{dx^2}+\cos x\frac{dy}{dx}=\delta\left(x-\frac{\pi}{2}\right)$. The value of $\frac{dy}{dx}$ at the point $x=\frac{\pi}{2}$

(a) is continuous (b) has a discontinuity of 3

(c) has a discontinuity of 1/3 (d) has a discontinuity of 1 **[CSIR June -2012]**

Soln. $\sin x\frac{d^2y}{dx^2}+\cos x.\frac{dy}{dx}=\delta\left(x-\frac{\pi}{2}\right)$

$$\Rightarrow \frac{d}{dx}\left[\sin x\frac{dy}{dx}\right]=\delta\left(x-\frac{\pi}{2}\right) \Rightarrow \int d\left[\sin x\frac{dy}{dx}\right]=\int\delta\left(x-\frac{\pi}{2}\right)dx$$

$$\Rightarrow \sin x\frac{dy}{dx}=\int\delta\left(x-\frac{\pi}{2}\right)dx$$

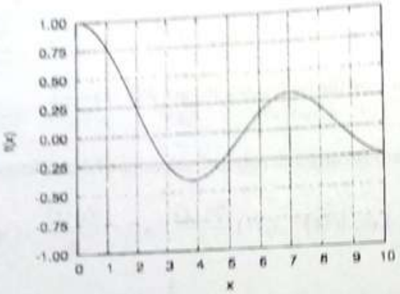
At $x=\frac{\pi}{2}$, the equation becomes $\frac{dy}{dx}=\int\delta\left(x-\frac{\pi}{2}\right)dx=1$

At $x\neq\frac{\pi}{2}$, the equation becomes $\frac{dy}{dx}=\frac{1}{\sin x}\int\delta\left(x-\frac{\pi}{2}\right)dx=0$

$\therefore \quad \frac{dy}{dx}$ has a discontinuity of 1 at $x=\frac{\pi}{2}$

Correct answer is (d)

5. The graph of the function f(x) shown below is best described by **[CSIR Dec -2012]**



(a) The bessel function $J_0(x)$ (b) $\cos x$

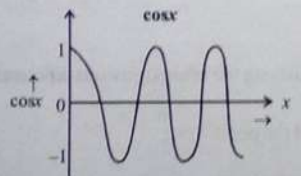
(c) $e^{-x}\cos x$ (d) $\frac{\cos x}{x}$

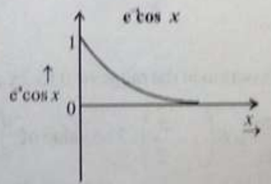
Soln. At x = 0, Bessel function $J_0(x)=1$

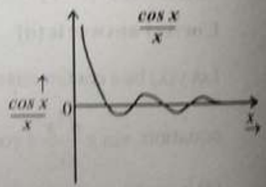
$J_0(x)$ has zeroes at x = 2.4048, 5.5201, 8.6537 .... etc.

So, the graph is best described by $J_0(x)$

The other functions are following plots.







Correct answer is (a).

6. A function $f(x)$ obeys the differential equation $\frac{d^2f}{dx^2}-(3-4i)f=0$ and satisfies the conditions $f(0)=1$ and $f(x)\to 0$ as $x\to\infty$. The value of $f(\pi)$ is : **[CSIR Dec -2012]**

(a) $e^{2\pi}$ (b) $e^{-2\pi}$ (c) $-e^{-2\pi}$ (d) $-e^{2\pi i}$

Soln. Given differential equation: $\frac{d^2f}{dx^2}-(3-4i)f=0$

Let, $f=c.e^{mx}$ be trial solution.

$$\Rightarrow \quad \frac{df}{dx}=c.m\,e^{mx} \quad \Rightarrow \quad \frac{d^2f}{dx^2}=c.m^2\,e^{mx}$$

Substituting in the given equation, we get

$$m^2-(3-4i)=0 \quad \left(e^{mx}\neq 0\right)$$

$$\Rightarrow \quad m=\sqrt{3-4i}=\sqrt{4-1-4i}=\sqrt{(2)^2+(i)^2-2.2i}=\sqrt{(2-i)^2}=\pm(2-i)$$

Therefore, $f(x)=A\,e^{(2-i)x}+B\,e^{-(2-i)x}$

Since, $f(0)=1 \quad \Rightarrow A+B=1$

Since, $f(x)\to 0$ when $x\to\infty \Rightarrow A=0$, then $B=1$

$$\therefore \quad f(x)=e^{-(2-i)x} \quad \Rightarrow f(\pi)=e^{-(2-i)\pi}=e^{-2\pi}e^{i\pi} \quad \Rightarrow \boxed{f(\pi)=-e^{-2\pi}}$$

Correct answer is (c).

7. Given that $\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}=e^{-t^2+2tx}$, the value of $H_4(0)$ is **[CSIR June -2013]**

(a) 12 (b) 6 (c) 24 (d) -6

Soln. $\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}=e^{-t^2+2tx}$

Putting $x=0$ in the above equation, we get

$$\sum_{n=0}^{\infty}H_n(0)\frac{t^n}{n!}=e^{-t^2}=1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\ldots\ldots$$

Comparing the coefficient of t^4 both sides, we get

$$\frac{H_4(0)}{4!}=\frac{1}{2!} \quad \Rightarrow H_4(0)=12$$

Correct answer is (a)

8. The solution of the differential equation $\frac{dx}{dt}=x^2$ with the initial condition $x(0)=1$ will blow up as t tend to **[CSIR June -2013]**

(a) 1 (b) 2 (c) 1/2 (d) ∞

Soln. Given: $\frac{dx}{dt}=x^2\Rightarrow\int\frac{dx}{x^2}=\int dt\Rightarrow-\frac{1}{x}=t+C$

Applying $x(0)=1$, we get $C=1\Rightarrow-\frac{1}{x}=t-1\Rightarrow x=\frac{1}{1-t}$

Therefore, the solution of the given differntial equation will blow up as t tends to 1.

Correct answer is (a)

9. Consider the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$$

[CSIR June -2014]

with the initial conditions $x(0)=0$ and $\dot{x}(0)=1$. The solution $x(t)$ attains its maximum value when 't' is

(a) 1/2 (b) 1 (c) 2 (d) ∞

Soln. $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$

Assume the trial solution : $x = ce^{mt}$

$$\Rightarrow \left(m^2+2m+1\right)c.e^{mt}=0 \Rightarrow \left(m^2+2m+1\right)=0 \quad \Rightarrow (m+1)^2=0 \quad \Rightarrow m=-1,-1$$

$$\therefore \quad x(t)=(c_1+c_2t)e^{-t}$$

Putting $x(0)=0 \quad \Rightarrow c_1=0$

$$\therefore \quad x(t)=c_2t\,e^{-t} \qquad \Rightarrow \dot{x}(t)=\frac{dx}{dt}=c_2e^{-t}-c_2te^{-t}$$

putting $\dot{x}(0)=1 \qquad \Rightarrow c_2=1$

$$\therefore \quad x(t)=t\,e^{-t} \qquad \Rightarrow \frac{dx}{dt}=e^{-t}-te^{-t} \qquad \Rightarrow \frac{d^2x}{dt^2}=te^{-t}-2e^{-t}$$

For maxima or minima, $\frac{dx}{dt}=0 \quad \Rightarrow t=1$

At $t=1, \frac{d^2x}{dt^2}=-e^{-1}=-ve$

Therefore, $x(t)$ attains its maximum value at $t=1$

Correct answer is (b)

10. Given,

$\sum_{n=0}^{\infty} P_n(x)t^n=\left(1-2xt+t^2\right)^{-1/2}$, for $|t|<1$, the value of $P_5(-1)$ is **[CSIR June -2014]**

(a) 0.26 (b) 1 (c) 0.5 (d) -1

Soln. $\left(1-2xt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty} P_n(x)t^n$

Putting $x=-1$ in the above equation, we get

$$\left(1+2t+t^2\right)^{-1/2}=\sum_{n=0}^{\infty} P_n(-1)t^n \Rightarrow (1+t)^{-1}=1-t+t^2-t^3+t^4-\ldots\ldots=\sum_{n=0}^{\infty} P_n(-1)t^n$$

Comparing the coefficient t^5 on the both sides, we get $P_5(-1)=-1$

Correct answer is (d)

11. The function $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$ satisfies the differential equation

[CSIR June -2014]

(a) $x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+(x^2+1)f=0$ (b) $x^2\frac{d^2f}{dx^2}+2x\frac{df}{dx}+(x^2-1)f=0$

(c) $x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+(x^2-1)f=0$ (d) $x^2\frac{d^2f}{dx^2}-x\frac{df}{dx}+(x^2-1)f=0$

Soln. The solution of the Bessel differential equation

$$x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+\left(x^2-k^2\right)f=0$$

is

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+k)!}\left(\frac{x}{2}\right)^{2n+k}$$

So, for the given function $f(x)$, k will be 1. Therefore, it satisfies the differential equation

$$x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+\left(x^2-1\right)f=0$$

Correct Option is (c)

GATE Previous Year Questions with Solutions

1. The solution of the system of differential equations, **[GATE 2001]**

$$\frac{dy}{dx}=y-z \text{ and } \frac{dz}{dx}=-4y+z$$

is given by (for A and B are arbitrary constants)

(a) $y(x)=Ae^{3x}+Be^{-x}; z(x)=-2Ae^{3x}+2Be^{-x}$

(b) $y(x)=Ae^{3x}+Be^{-x}; z(x)=2Ae^{3x}+2B^{-x}$

(c) $y(x)=Ae^{3x}+Be^{-x}; z(x)=2Ae^{3x}-2Be^{-x}$

(d) $y(x)=Ae^{3x}+Be^{-x}; z(x)=-2Ae^{3x}-2Be^{-x}$

Soln. $\frac{dy}{dx}=y-z$; $\frac{dz}{dx}=-4y+z$

Equations can be expressed as

$$(D-1)y+z=0 \qquad (1)$$

$$(D-1)z+4y=0 \qquad (2)$$

$$\begin{array}{ll} (1)\times(D-1) & \Rightarrow (D-1)^2y+(D-1)z=0 \\ \underline{(2)\times 1} & \underline{\Rightarrow 4y \qquad +(D-1)z=0} \end{array}$$

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3y=0$$

$$\Rightarrow y=Ae^{3x}+Be^{-x}$$

So, $z=y-\frac{dy}{dx}=\left(Ae^{3x}+Be^{-x}\right)-\left(3Ae^{3x}-Be^{-x}\right)=-2Ae^{3x}+2Be^{-x}$

Correct option is (a)

[GATE 2002]

2. The solution of the differential equation,

$$(1+x)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}-y(x)=0 \text{ is}$$

(a) Ax^2+B (b) $Ax+Be^{-x}$ (c) $Ax+Be^{x}$ (d) $Ax+Bx^2$

where A and B are constants.

Soln. $y=Ax+Be^{-x}$ $\Rightarrow \frac{dy}{dx}=A-Be^{-x}$ $\Rightarrow \frac{d^2y}{dx^2}=Be^{-x}$

$$(1+x)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y$$

$$=(1+x)\left(Be^{-x}\right)+x\left(A-Be^{-x}\right)-\left(Ax+Be^{-x}\right)$$

$$=Be^{-x}+xBe^{-x}+Ax-xBe^{-x}-Ax-Be^{-x}=0$$

Correct option is (b)

3. Consider the differential equation $\frac{d^2x}{dt^2}+\frac{2dx}{dt}+x=0$. At time $t=0$, it is given that $x=1$ and $\frac{dx}{dt}=0$. At $t=1$, the value of x is given by [GATE 2003]

(a) $\frac{1}{e}$ (b) $\frac{2}{e}$ (c) 1 (d) $\frac{3}{e}$

Soln. $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$

Let, $x=C.e^{mt}$ be the trial solution.

So, $\left(m^2+2m+1\right)C.e^{mt}=0 \Rightarrow m=-1,-1$

Therefore, $x=\left(C_1+C_2t\right)e^{-t}$

At $t=0, x=1$ then $C_1=1$; so, $x=\left(1+C_2t\right)e^{-t}$

Now, $\frac{dx}{dt}=C_2e^{-t}+\left(1+C_2t\right)\left(-e^{-t}\right)$

At $t=0, \frac{dx}{dt}=0$, then $C_2=-1=0 \Rightarrow C_2=1$

So, $C_2x=\left(1+t\right)e^{-t}$; At $t=1, x=\frac{2}{e}$

Correct option is (b)

Linked Answer type Q.4 and Q.5

For the differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

4. One of the solution is [GATE 2005]

(a) e^x (b) $\ln x$ (c) e^{-x^2} (d) e^{x^2}

5. The second linearly independent solution is

(a) e^{-x} (b) xe^x (c) x^2e^x (d) x^2e^{-x}

Soln. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

Let the trial solution by $y=C.e^{mx}$

So, $e^{mx}\left(m^2-2m+1\right)=0$

$\Rightarrow (m-1)^2=0$ $\left(\text{since, } e^{mx}\neq 0\right)$

The solution will be $y=\left(c_1+c_2x\right)e^{mx}=\left(c_1+c_2x\right)e^x$

Correct option is (a) and (b)

6. The points, where the series solution of the Legendre differential equation $\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\frac{3}{2}\left(\frac{3}{2}+1\right)y=0$ [GATE 2007]

(a) 0 and 1 (b) 0 and −1 (c) −1 and 1 (d) $\frac{3}{2}$ and $\frac{5}{2}$

Soln. The solution of Legendre differential equation will be

$$y=A\,P_n(x)+BQ_n(x)$$

where $P_n(x)$ is Legendre Polynomial of order 'n' and $Q_n(x)$ is Legendre function of second kind and given by

$$Q_0(x)=\frac{1}{2}\ell n\left(\frac{1+x}{1-x}\right)$$

$$Q_1(x)=\frac{x}{2}\ell n\left(\frac{1+x}{1-x}\right)-1$$

This shows that, $Q_n(x)$ will diverge at $x=\pm 1$.

Correct option is (c)

7. Solution of the differential equation $x\frac{dy}{dx}+y=x^4$, with the boundary condition that $y=1$ at $x=1$, is [GATE 2008]

(a) $y=5x^4-4$ (b) $y=\frac{x^4}{5}+\frac{4x}{5}$ (c) $y=\frac{4x^4}{5}+\frac{1}{5x}$ (d) $y=\frac{x^4}{5}+\frac{4}{5x}$

Soln. $x\frac{dy}{dx}+y=x^4$ $\Rightarrow \frac{dy}{dx}+\frac{y}{x}=x^3$

$I.F.=e^{\int dx/x}=e^{\ln x}=x$

Solution will be $xy=\int x^3x\,dx$ $\Rightarrow xy=\frac{x^5}{5}+c$

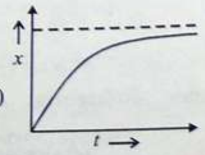
Putting the condition, at $x=1, y=1$, we get $c=\frac{4}{5}$

So, $xy=\frac{x^5}{5}+\frac{4}{5} \quad \Rightarrow y=\frac{x^4}{5}+\frac{4}{5x}$

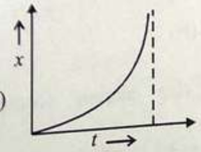
Correct option is (d)

8. Which one of the following curves gives the solution of the differential equation $k_1\frac{dx}{dt}+k_2x=k_3$, where k_1, k_2 and k_3 are positive constant with initial conditions $x=0$ and $t=0$?

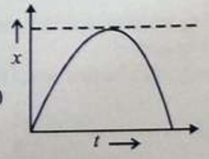
(a)



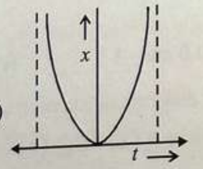
(b)



(c)



(d)



Soln. $k_1\frac{dx}{dt}+k_2x=k_3 \Rightarrow \frac{dx}{dt}+\frac{k_2}{k_1}x=\frac{k_3}{k_1}$

$I.F.=e^{\int\frac{k_2}{k_1}dt}=e^{(k_2/k_1)dt}$

Therefore, the solution will be

$$x\cdot e^{(k_2/k_1)t}=\int\frac{k_3}{k_1}\exp\left(\frac{k_2}{k_1}t\right)dt$$

$$\Rightarrow \quad x\cdot e^{(k_2/k_1)t}=\frac{k_3}{k_2}\exp\left(\frac{k_2}{k_1}t\right)+c$$

$$\Rightarrow \quad x=\frac{k_3}{k_2}+c\cdot\exp\left(-\frac{k_2}{k_1}t\right)$$

Putting the condition, at $t=0, x=0$, we get $C=-\frac{k_3}{k_2}$

Therefore, $x=\frac{k_3}{k_2}\left[1-\exp\left(-\frac{k_2}{k_1}t\right)\right]$

Correct option is (a)

---

9. The solution of the differential equation for $y(t): \frac{d^2y}{dt^2}-y=2\cosh(t)$, subject to the initial conditions $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=0$ is

[GATE 2010]

(a) $\frac{1}{2}\cosh(t)+t\sinh(t)$

(b) $-\sinh(t)+t\cosh(t)$

(c) $t\cosh(t)$

(d) $t\sinh(t)$

Soln. $\frac{d^2y}{dt^2}-y=2\cdot\cosh t$

$C.F.=Ae^t+Be^{-t}$

$P.I.=\frac{1}{D^2-1}(2\cdot\cosh t)=\frac{1}{(D^2-1)}(e^t+e^{-t})$

$=t\frac{1}{2D}(e^t+e^{-t})=\frac{t}{2}(e^t-e^{-t})=t\sinh t$

Total solution, $y=Ae^t+Be^{-t}+t\sinh t$

Putting the condition, $\left.\frac{dy}{dt}\right|_{t=0}=0 \Rightarrow A-B=0$

Solving above two equations, $A=0, B=0$

Therefore, $y=t\sin t$

Correct option is (d)

10. Given the recurrence relation for the Legendre polynomials

$$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)$$

which one of the following integrals has a non-zero value?

(a) $\int_{-1}^{+1}x^2P_n(x)P_{n+1}(x)dx$

(b) $\int_{-1}^{+1}xP_n(x)P_{n+2}(x)dx$

(c) $\int_{-1}^{+1}x[P_n(x)]^2dx$

(d) $\int_{-1}^{+1}x^2P_n(x)P_{n+2}(x)dx$

Soln. $\int_{-1}^{1}x^2P_n(x)P_{n+2}(x)dx=\int_{-1}^{1}[xP_n(x)][xP_{n+2}(x)]dx$

$=\int_{-1}^{1}\left[\frac{(n+1)}{(2n+1)}P_{n+1}(x)+\frac{n}{(2n+1)}P_{n-1}(x)\right]\left[\frac{(n+3)}{(2n+5)}P_{n+3}(x)+\frac{(n+2)}{(2n+5)}P_{n+1}(x)dx\right]$

$=\int_{-1}^{1}\frac{(n+1)(n+2)}{(2n+1)(2n+5)}P_{n+2}(x)P_{n+1}(x)dx$

[Using the orthonormal property, $\int_{-1}^{1}P_m(x)P_n(x)dx=\frac{2}{2n+1}\delta_m$]

$$=\frac{(n+1)(n+2)}{(2n+1)(2n+5)}\frac{2}{(2n+3)}\neq 0$$

Correct option is (d)

11. The solutions to the differential equation $\frac{dy}{dx}=-\frac{x}{y+1}$ are a family of [GATE-2011]

(a) Circles with different radii
(b) Circles with different centres.
(c) Straight lines with different slopes
(d) Straight lines with different intercepts on the y-axis

Soln. $\frac{dy}{dx}=-\frac{x}{y+1}\Rightarrow\int(y+1)\,dy=-\int x\,dx$

$\Rightarrow \frac{y^2}{2}+y=-\frac{x^2}{2}+c$

$\Rightarrow x^2+y^2+2y=2c$

$\Rightarrow x^2+(y+1)^2=2c+1$

This represents a family of circles having $(0,-1)$ and radius of $\sqrt{2c+1}$ units.

Correct option is (a)

12. The solution of the differential equation $\frac{d^2y}{dt^2}-y=0$ subject to the boundary conditions $y(0)=1$ and $y(\infty)=0$, is [GATE-2014]

(a) $\cos t+\sin t$ (b) $\cosh t+\sinh t$ (c) $\cos t-\sin t$ (d) $\cosh t-\sinh t$

Soln. $\frac{d^2y}{dt^2}-y=0$

Assume, the trial solution to be $y=c\cdot e^{mx}$

$\Rightarrow m^2-1=0\Rightarrow m=\pm1$

So, $y=Ae^{t}+Be^{-t}$

Putting $y(t=0)=1$, we get $A+B=1$

Putting $y(t=\infty)=0$, we get $A=0, B=1$

Therefore, $y(t)=e^{-t}=\cosh t-\sinh t$

Correct option is (d)

13. A function $y(z)$ satisfies the ordinary differential equation $y''+\frac{1}{z}y'-\frac{m^2}{z^2}y=0$, where $m=0,1,2,3,\ldots$

Consider the four statements P, Q, R, S as given below. [GATE 2015]

P: z^m and z^{-m} are linearly independent solutions for all values of m

Q: z^m and z^{-m} are linearly independent solutions for all values of $m>0$

R: $\ell n\, z$ and 1 are linearly independent solutions for $m=0$

S: z^m and $\ell n\, z$ are linearly independent solutions for all values of m

The correct option for the combination of valid statement is

(a) P, R and S only (b) P and R only (c) Q and R only (d) R and S only

Soln. $y''+\frac{1}{z}y'-\frac{m^2}{z^2}y=0$

$\Rightarrow z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}-m^2y=0\rightarrow$ Cauchy Euler equation.

Putting $z=e^x$ and replacing

$z\frac{dy}{dz}$ by $\frac{dy}{dx}$ and $z^2\frac{d^2y}{dz^2}$ by $\frac{d^2y}{dx^2}-\frac{dy}{dx}$

We get,

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}+\frac{dy}{dx}-m^2y=0$$

Trial soln : $y=C\cdot e^{kx}$

$\Rightarrow (k^2-m^2)=0\Rightarrow k=\pm m$

So, $y=C_1e^{mx}+C_2e^{-mx}=C_1z^m+C_1z^{-m}$

Therefore, z^m and z^{-m} are linearly independent solutions for $m>0$.

For $m=0, k=0\Rightarrow y=(C_1+C_2x)e^{0\cdot x}=(C_1+C_2\ell nz)$

Therefore, 1 and $l$nz are linearly independent solutions for $m=0$

Correct option is (c)

IIT-JAM Previous Year Questions with Solutions

1. The solution of the differnetial equation $\mathrm{dz}(\mathrm{x,y})+\mathrm{xz}(\mathrm{x,y})\mathrm{dx}+\mathrm{yz}(\mathrm{x,y})\mathrm{dy}=0$ ....

Soln. $dz(x,y)+xz(x,y)dx+yz(x,y)dy=0$

$\Rightarrow dz=-z(xdx+ydy)\ \Rightarrow\int\frac{dz}{z}=-\int xdx-\int ydy$

$\Rightarrow \ln z=-\frac{x^2}{2}-\frac{y^2}{2}+\ln A$

$\Rightarrow z(x,y)=A\,e^{-(x^2+y^2)/2}$

2. Solve the differential equation $xy\frac{dy}{dx}=3y^2+x^2$ with the initial condition y = 2 when x = 1.

Soln. $xy\frac{dy}{dx}=3y^2+x^2\quad\Rightarrow\frac{dy}{dx}=\frac{3y^2+x^2}{xy}$ ... (i)

putting $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$

from (i), $v+x\frac{dv}{dx}=\frac{3v^2x^2+x^2}{vx^2}=\frac{3v^2+1}{v}$

$\Rightarrow x\frac{dv}{dx}=\frac{3v^2+1}{v}-v=\frac{2v^2+1}{v}$

$\Rightarrow \quad \int \frac{v}{2v^2+1} dv = \int \frac{dx}{x}$ (putting $2v^2+1=t \Rightarrow 4vdv = dt$)

$\Rightarrow \quad \frac{1}{4}\int \frac{dt}{t} = \int \frac{dx}{x} \quad \Rightarrow \ln t = \ln x^4 + \ln c$

$\Rightarrow \quad t = cx^4 \Rightarrow 2v^2+1 = cx^4 \Rightarrow 2y^2 + x^2 = cx^6$

Putting $y=2$ when $x=1$, we get $c=9$

Solution : $2y^2 + x^2 = 9x^6$

3. Find the solution of the differential equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$ with the boundary condition $y(0) = 2$ and $\left.\frac{dy}{dx}\right|_{x=0} = 2$, giving all steps clearly. Find the value of x where $y=0$.

Soln. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$

Let us assume that trial solution by $y = C.e^{mx}$

Putting the trial solution in the given differential equation. We get,

$\Rightarrow \quad (m^2+5m)C.e^{mx} = 0$

$\Rightarrow \quad (m^2+5m) = 0$ [For non-trivial solution]

$\Rightarrow \quad m = 0, -5$

So, $y = C_1.e^{0.x} + C_2e^{-5x} = C_1 + C_2e^{-5x}$

Putting $y = 2$ at $x = 0$, $\Rightarrow C_1 + C_2 = 2$

Now, $\frac{dy}{dx} = -5C_2e^{-5x}$

Putting $\frac{dy}{dx} = 2$ at $x = 0 \Rightarrow -5C_2 = 2 \Rightarrow C_2 = \frac{-2}{5}$

So, $C_1 + C_2 = 2 \Rightarrow C_1 = \frac{12}{5}$

The solution of the given differential equation will be

$$\boxed{y = \frac{12}{5} - \frac{2}{5}e^{-5x}}$$

4. Consider the equation $\frac{dy}{dx} = \frac{y^2}{x}$ with the boundary condition $y(1) = 1$. Out of the following, the range of x in which y is real and finite, is

(a) $-\infty \le x \le -3$ (b) $-3 \le x \le 0$ (c) $0 \le x \le 3$ (d) $3 \le x \le \infty$

Soln. $\frac{dy}{dx} = \frac{y^2}{x}$

$\Rightarrow \quad \int \frac{dy}{y^2} = \int \frac{dx}{x} \quad \Rightarrow -\frac{1}{y} = \ln x + c$

Applying the condition, $y(1) = 1$

We get, $c = -1$

Therefore, $-\frac{1}{y} = \ln x - 1 \Rightarrow y = \frac{1}{1-\ln x}$

Now, at $x = e = 2.718$, $y = \infty$

Therefore, for $3 \le x \le \infty$, the value of y will be real and finite.

Correct option is (d)

PRACTICE SET

1. An integrating factor I(x) is found for the differential equation $\left(1+x^2\right)\frac{dy}{dx}+xy=0$ and the equation is rewritten as $\frac{d}{dx}\left[I(x)y\right]=0$. Which of the following option is correct?

(a) $I(x)=e^{x^2/2}$ (b) $I(x)=\sqrt{1+x^2}$ (c) $I(x)=1+x^2$ (d) $I(x)=0$

2. Find the general solution to $x^2\frac{dy}{dx}+2xy=\cos^2 x$ is:

(a) $y=\frac{1}{2x}+\frac{\sin 2x}{4x^2}+\frac{c}{x^2}$ (b) $y=\frac{1}{2x}+\frac{\sin 2x}{4x^2}+c$

(c) $y=\frac{\sin^2 x+c}{x^2}$ (d) $y=\frac{x}{2}+\frac{\sin 2x}{4}+c$

3. The general solution to $\frac{d^2z}{dt^2}+6\frac{dz}{dt}+9z=0$ is z. Which of the following options are correct?

(a) As $t\to\infty, z\to A$ for any value of B (b) The behaviour of z as $t\to\infty$ depends on A, B

(c) As $t\to\infty, z\to 0$ for any value of A and B (d) As $t\to\infty, z\to\infty$ for any value of A and B.

4. The particular solution to $\frac{d^2y}{dt^2}-3\frac{dy}{dt}-4y=0$ satisfying $y(0)=0$ and $y'(0)=5$

(a) $y=e^{t}-e^{-4t}$ (b) $y=e^{-t}-e^{4t}$ (c) $y=e^{-4t}-e^{t}$ (d) $y=e^{4t}-e^{-t}$

5. The general solution of differential equation $4x^2y''-8xy'+9y=0$ is:

(a) $C_1 e^{5x/2}+C_2 e^{-3x/2}$ (b) $C_1 e^{3x/2}+C_2 e^{-3x/2}$ (c) $(C_1+C_2\log x)x^{3/2}$ (d) $C_1 x^{3/2}+C_2 x^{-3/2}$

6. The particular integral of $\frac{d^4y}{dx^4}-y=\cos x\cosh x$ is:

(a) $y=-\frac{1}{5}\sinh x\cos x$ (b) $y=-\frac{1}{5}\cosh x\cos x$ (c) $y=-\frac{1}{5}\sinh x\sin x$ (d) $y=-\frac{1}{5}\cosh x\sin x$

7. The differential equation of all circles passing through the origin and having their centres on the y-axis is

(a) $\left(x^2+y^2\right)dy/dx=2xy$ (b) $\left(x^2-y^2\right)dy/dx=2xy$

(c) $dy/dx=2xy\left(x^2+y^2\right)$ (d) $dy/dx=2xy\left(x^2-y^2\right)$

8. The geometrical interpretation of the graph of the differential equation: $\frac{dy}{dx}=-\frac{x}{y}$ is a family of

(a) Hyperbola (b) Ellipse (c) Parabola (d) Circle

9. If f_1 and f_2 be the integrating factors of the differential equations $x\frac{dy}{dx}+2y=1$ and $\frac{dy}{dx}-\frac{2}{x}y=\frac{1}{x}$ respectively, then

(a) $f_1=x^4f_2$ (b) $f_1=x^3f_2$ (c) $f_1=x^2f_2$ (d) $f_1=f_2$

10. Which of the following transformations reduces the differential equation $\frac{dy}{dx}-\frac{\tan y}{1-x}=(1+x)e^x\sec y$ into the form $\frac{dy}{dx}+P(x)z=Q(x)$, by the substitution

(a) $z=\sin y$ (b) $z=\cos y$ (c) $z=\operatorname{cosec} y$ (d) $z=\tan y$

11. The bessel function $J_{-2}(x)$ can be written as

(a) $J_1(x)$ (b) $\frac{d}{dx}(J_1(x))$ (c) $J_2(x)$ (d) $\int J_1(x)dx$

12. The expression $\frac{d}{dx}\left(x^3J_8(x)\right)$ can be simplified to

(a) $x^2J_2(x)$ (b) $x^3J_4(x)$ (c) $x^3J_2(x)$ (d) $-x^3J_2(x)$

13. The bessel function $J_0(1)$ can be best described by the series:

(a) $1+\frac{1}{2^2}+\frac{1}{2^2.4^2}+\frac{1}{2^2.4^2.6^2}+\dots\dots\dots$ (b) $1+\frac{1}{2^2}+\frac{1}{2^2.4^2}+\frac{1}{2^2.4^2.6^2}+\dots\dots\dots$

(c) $1+\frac{1}{2.2^2}+\frac{1}{3!.2^2.4^2}+\frac{1}{4!.2^2.4^2.6^2}+\dots\dots\dots$ (d) $1-\frac{1}{1^2}+\frac{1}{1^2.3^2}-\frac{1}{1^2.3^2.5^2}+\frac{1}{1^2.3^2.5^2.7^2}-\dots\dots\dots$

14. The value of the integral $\int_0^1 x\left[J_1(x)\right]^2 dx$ is equal to

(a) $J_2(1)^2$ (b) 0 (c) $\frac{1}{2}\left[J_2(1)\right]^2$ (d) $J_1(1)^2$

15. The bessel function $J_2(x)$ can be written as

(a) $J_1(x)-J_0(x)$ (b) $2J_1(x)-J_0(x)$ (c) $\frac{2}{x}J_1(x)-J_0(x)$ (d) $\frac{2}{x^2}J_1(x)-J_0(x)$

16. The expression $2J'_1(x)$ can be simpified to

(a) $J_0(x)-J_2(x)$ (b) $J_1(x)-J_0(x)$ (c) $J_0(x)-J_3(x)$ (d) $J_1(x)-J_3(x)$

17. The co-efficient of z^{-3} in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ is equal to

(a) $J_3(x)$ (b) $J_3(0)$ (c) $J_3(-x)$ (d) $-J_3(x)$

18. The function $f(x)=x^2$ can be written in terms of Legendre Polynomials as

(a) $\frac{2P_2(x)+P_0(x)}{3}$ (b) $\frac{2P_1(x)-P_2(x)}{3}$ (c) $\frac{P_2(x)-P_1(x)}{3}$ (d) $\frac{P_2(x)-P_0(x)}{3}$

19. The function $f(x)=x^2-x$ can be written in terms of Legendre Polynomials as

(a) $\frac{3P_2(x)-P_1(x)+P_0(x)}{2}$ (b) $\frac{2P_2(x)-3P_1(x)+P_0(x)}{3}$

(c) $\frac{2P_2(x)-P_1(x)-P_0(x)}{3}$ (d) $\frac{2P_2(x)-P_1(x)+P_0(x)}{3}$

20. The solution of the differential equation $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+6y=0$ is

(a) $P_n(x)$ (b) $P_1(x)$ (c) $P_2(x)$ (d) $P_3(x)$

21. The value of the integral $\int_{-1}^{1}[P_3(x)]^2\,dx$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{2}{5}$ (c) $\frac{2}{7}$ (d) $\frac{1}{5}$

22. The value of the integral $\int_{-1}^{1}P_n(x)\,dx$ is equal to

(a) 0 (b) 1 (c) $\frac{1}{2n+1}$ (d) $\frac{2}{n^2-1}$

23. The value of the integral $\int_{0}^{1}P_{2n}(x)\,dx$ is equal to

(a) 0 (b) 1 (c) $\frac{1}{2n+1}$ (d) $\frac{2}{2n+1}$

24. The value of the integral $\int_{-1}^{1}P_n(x)\left(1-2xz+z^2\right)^{-\frac{1}{2}}dx$ is equal to

(a) $\frac{2}{2n+1}$ (b) $\frac{2z^n}{2n+1}$ (c) $\frac{z^n}{2n+1}$ (d) 0

25. The expression $P'_{2n}(0)$ can be written as

(a) 0 (b) 1 (c) $\frac{1}{2n}$ (d) $\frac{(-1)^n.n!}{2^n}$

26. Solve the differential equations: (Separation of variables):

(i) $x\sqrt{1+y^2}\,dx+y\sqrt{1+x^2}\,dy=0$ given that $y(0)=\sqrt{3}$

(ii) $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0$ given that $y\left(\frac{\pi}{3}\right)=0$

(iii) $3\tan y.e^x\,dx+(1-e^x)\sec^2 y\,dy=0$ given that $y(1)=\frac{\pi}{4}$

(iv) $\frac{dy}{dx}=(4x+y+1)^2$ given that $y(0)=1$

(v) $x^4\frac{dy}{dx}+x^3y=-\sec(xy)$ given that $y(1)=\frac{\pi}{6}$

(vi) $\left(2x^2+3y^2-7\right)x\,dx-\left(3x^2+2y^2-8\right)y\,dy=0$ given that $y(1)=1$

[*Ans.* (*i*) $\sqrt{1+x^2}+\sqrt{1+y^2}=3$; (*ii*) $\sin x\left(e^y+1\right)=\sqrt{3}$; (*iii*) $\left(1-e^x\right)^3=(1-e)^3\tan y$

(*iv*) $\tan^{-1}\left(\frac{4x+y+1}{2}\right)=2x+\frac{\pi}{4}$; (*v*) $\sin(xy)=\frac{1}{2x^2}$; (*vi*) $x^2+y^2-3=\left(x^2-y^2-1\right)^5$]

27. Solve the differential equations: (Homogeneous type)

(i) $(y^2-xy)dx+x^2dy=0$ given that $y(1)=1$

(ii) $\left(y\sin\frac{y}{x}-x\right)dx-x\sin\frac{y}{x}dy=0$ given that $y(4)=\pi$

(iii) $(1+e^{x/y})dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$ given that $y(0)=1$

(iv) $x\frac{dy}{dx}=y\left(\log\frac{y}{x}+1\right)$ given that $y(1)=2$

[*Ans.* (*i*) $\frac{x}{y}=\ln x+1$; (*ii*) $\cos\left(\frac{y}{x}\right)=\frac{1}{\sqrt{2}}-\ln 4+\ln x$; (*iii*) $e^{x/y}+\frac{x}{y}=\frac{1}{y}$; (*iv*) $\ln\frac{y}{x}=x\ln 2$]

28. Solve the differential equations: (Linear type)

(i) $\cos^2 x\frac{dy}{dx}+y=\tan x$ given that $y\left(\frac{\pi}{4}\right)=1$

(ii) $(1+y^2)dx=(\tan^{-1}y-x)dy$ given that $x(1)=-1$

(iii) $(x^3-x)\frac{dy}{dx}-(3x^2-1)y=x^5-2x^3+x$ given that $y(e)=e$

(iv) $(1-x^2)\frac{dy}{dx}+2xy=x(1-x^2)^{1/2}$ given that $y(0)=1$

(v) $3\frac{dy}{dx}+3\frac{y}{x}=2x^4y^4$ given that $y(1)=1$

(vi) $\cos x\frac{dy}{dx}+4y\sin x=4\sqrt{y}\sec x$ given that $y\left(\frac{\pi}{4}\right)=1$

(vii) $r\sin\theta-\frac{dr}{d\theta}\cos\theta=r^2$ given that $r\left(\frac{\pi}{4}\right)=1$

[*Ans.* (*i*) $y=\left(\tan x-1\right)+e^{(1-\tan x)}$; (*ii*) $x=\tan^{-1}y-1-\frac{\pi}{4}e^{\left(\frac{\pi}{4}-\tan^{-1}y\right)}$; (*iii*) $\frac{y}{x^3-x}=\ln x-\frac{e^2}{e^2-1}$;

(*iv*) $y=\sqrt{1-x^2}$; (*v*) $\frac{1}{y^3}=-x^5+2x^3$; (*vi*) $\sec^2 x.\sqrt{y}=2\left(\tan x+\frac{1}{3}\tan^3 x\right)-\frac{2}{3}$;

(*vii*) $\frac{1}{r}\sec\theta=\tan\theta+\left(\sqrt{2}-1\right)$]

29. Solve the following differential equations: (exact equation)

(i) $\left(x^4-2xy^2+y^4\right)dx-\left(2x^2y-4xy^3+\sin y\right)dy=0$ given that $y(1)=0$

(ii) $\left[1+\ln(xy)\right]dx+\left[1+\frac{x}{y}\right]dy=0$ given that $y(1)=1$

[*Ans.* (*i*) $\frac{x^5}{5}-x^2y^2+xy^4+\cos y=\frac{6}{5}$; (*ii*) $x\ln(xy)+y=1$]

30. A inductance 'L' and resistance 'R' are connected in seriers with an e.m.f 'E'. If the current is zero at time t =
0, find the current as a function of time. Also find the current at the end of 0.01 sec. If E = 100Volts, L=100
H, R = 20 Ohm.

A particle falls freely under gravity from a great height, there being a resistance proportional to the velocity of
31. A particle falls freely under gravity from a great height, there being a resistance proportional to the velocity of
fall. Find the velocity and displacement as function of time.

32. The rate at which a hot body cools is proportional to the difference in temparature between it and surroundings
(**Newton's law of Cooling**). A body is heated to 110^0C and placed in air at 10^0. It's temparature becomes
60^0 after 1 hour. After what additional time it cool down to 30^0?

33. In one dimensional steady state heat coduction for a hollow cyllinder with thermal conductivity 'k' in the region $a \le r \le b$, the temparature T_r at a distance r ($a \le r \le b$) is given by $\frac{d}{dr}\left[r\frac{dT_r}{dr}\right]=0$, subjected to the conditions $T_r = T_1$ at $r = a$ and $T_r = T_2$ at $r = b$. Determine the steaady state temaparature distributuion T_r.

[Ans. (5) $i=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right); i=5\left(1-e^{-0.002}\right)$, (6) $x=\frac{gm}{k}\left(t+\frac{m}{k}\left(e^{-(k/m)t}-1\right)\right)$,

(7) $\left(\frac{\ln 5}{\ln 2}-1\right)hrs.$, (8) $T_r=\frac{T_1-T_2}{\ln\left(\frac{r_1}{r_2}\right)}\ln r+\frac{T_2\ln r_1-T_1\ln r_2}{\ln\left(\frac{r_1}{r_2}\right)}$]

34. Solve the differential equations: (Homogeneous type):

(i) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ given that $y(0)=-1, y'(0)=0$

(ii) $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$ given that $y(0)=2, y'(0)=1$

(iii) $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=0$ given that $y(0)=1, y'(0)=2$

(iv) $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}+4\frac{dy}{dx}-8y=0$ given that $y(\infty)=0, y'(0)=2i$

[*Ans.* (*i*) $y=2e^{3x}-3e^{2x}$; (*ii*) $y=e^{-3x}\left(2\cos 2x+\frac{7}{2}\sin 2x\right)$; (*iii*) $y=(1-2x)e^{4x}$; (*iv*) $y=-e^{-2ix}$]

35. Solve the differential equations: (Non-homogeneous type):

(i) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=e^x\cosh 2x$

(ii) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=\sinh x$

(iii) $2\frac{d^2y}{dx^2}+3\frac{dy}{dx}+4y=x^2-2x$

(iv) $\left(D^2-4D+3\right)y=x^3$

(v) $\frac{d^2y}{dx^2}+4y=2\cos x\cos 3x$

(vi) $\frac{d^3y}{dx^3}+y=\cos^2\left(\frac{x}{2}\right)+e^{-x}$

(vii) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=x^3e^{2x}$

(viii) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$

(ix) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x\sin x$

(x) $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{3x}$

[*Ans.* (*i*) $y=c_1e^{3x}+c_2e^{-2x}+\frac{1}{10}xe^{3x}-\frac{1}{8}e^{-x}$; (*ii*) $y=e^{-x}\left(c_1\cos x+c_2\sin x\right)+\frac{1}{10}e^{x}+\frac{1}{2}e^{-x}$;

(*iii*) $y=e^{-\frac{3}{4}x}\left(c_1\cos\frac{\sqrt{23}}{4}x+c_2\sin\frac{\sqrt{23}}{4}x\right)+\frac{1}{32}\left(8x^2-28x+13\right)$; (*iv*) $y=c_1e^{x}+c_2e^{3x}+$
$\frac{1}{27}\left(9x^3+36x^2+78x+80\right)$; (*v*) $y=c_1\cos 2x+c_2\sin 2x-\frac{1}{12}\cos 4x+\frac{x}{4}\sin 4x$;

(*vi*) $y=c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)+\frac{1}{2}+\frac{1}{4}\left(\cos x-\sin x+e^{-x}\right)$;

(*vii*) $y=\left(c_1+c_2x\right)e^{2x}+e^{2x}\frac{x^5}{20}$; (*viii*) $y=\left(c_1+c_2x\right)e^{x}-e^{x}\left(x\sin x+2\cos x\right)$;

(*ix*) $y=\left(c_1+c_2x\right)e^{x}+\frac{1}{2}x\cos x+\frac{1}{2}\cos x-\frac{1}{2}\sin x$; (*x*) $y=c_1e^{x}+c_2e^{3x}+c_3e^{-2x}+\frac{x}{10}e^{3x}$]

36. Solve the following differential equations:

(i) $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$

(ii) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\sin\left(\ln x^2\right)$

(iii) $\frac{d^2y}{dx^2}+y=\sec x\tan x$

(iv) $\frac{d^2y}{dx^2}+y=\frac{2}{1+e^x}$

(v) $\frac{dx}{dt}+4x+3y=t, \frac{dy}{dt}+2x+5y=e^t$

(vi) $\frac{dx}{dt}+2y+\sin t=0, \frac{dy}{dt}-2x-\cos t=0$

[*Ans.* (*i*) $y=c_1x^4+\frac{c_2}{x}+\frac{(\ln x)}{5}x^4$; (*ii*) $y=c_1\cos(\ln x)+c_2\sin(\ln x)-\frac{1}{3}\sin\left(\ln x^2\right)$;

(*iii*) $y=c_1\cos x+c_2\sin x+x\cos x+\sin x\ln(\sec x)-\sin x$;

(*iv*) $y=c_1e^{x}+c_2e^{-x}-1+e^{x}\ln\left(e^{-x}+1\right)-e^{-x}\ln\left(e^{x}+1\right)$;

(*v*) $x=c_1e^{-2t}+c_2e^{-7t}+\frac{5}{14}t-\frac{31}{196}-\frac{1}{8}e^{t}, y=-\frac{2}{3}c_1e^{-2t}+c_2e^{-7t}-\frac{1}{7}t+\frac{9}{98}+\frac{5}{24}e^{t}$;

(*vi*) $x=c_1\cos 2t+c_2\sin 2t-\cos t, y=c_1\sin 2t-c_2\cos 2t-\sin t$]

37. The radial displacement 'u' in a rotating disc at a distance 'r' from the axis is given by,

$$r^2\frac{d^2u}{dr^2}+r\frac{du}{dr}-u+kr^3=0 \quad (k \text{ is constant})$$

Find the displacement 'u' as a function of 'r' subject to the conditions $u(0)=0, u(a)=0$.

38. Check whether the following functions are linearly dependent or not?

(i) e^{ax}, e^{bx}

(ii) $e^{ax}\sin bx, e^{ax}\cos bx$

39. Find the singular points and discuss their nature for the following differential equation.

(i) $2x^2y''+3xy'+(x^2-4)y=0$ (ii) $x(x-2)^2y''+2(x-2)y'+(x+3)y=0$

40. Express the following polynomials into legendre polynomials:

(i) $4x^3+6x^2+7x+2$ [**Ans.** $\frac{8}{5}P_3(x)+4P_2(x)+\frac{47}{5}P_1(x)+4P_0(x)$]

(ii) $4x^3-2x^2-3x+8$ [**Ans.** $\frac{8}{5}P_3(x)-\frac{4}{3}P_2(x)-\frac{3}{5}P_1(x)+\frac{22}{3}P_0(x)$]

(iii) $x^4+2x^3-6x^2+5x-3$ [**Ans.** $\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)-\frac{24}{7}P_2(x)+\frac{31}{5}P_1(x)-\frac{24}{5}P_0(x)$]

(iv) x^5 [**Ans.** $\frac{8}{63}P_5(x)+\frac{7}{9}P_3(x)+\frac{3}{7}P_1(x)$]

41. Prove that:

(i) $\sum_{n=0}^{\infty}P_n(x)=\frac{1}{\sqrt{2-2x}}$ (ii) $\frac{1-z^2}{\left(1-2xz+z^2\right)^{3/2}}=\sum_{n=0}^{\infty}(2n+1)P_n(x)$

(iii) $\log\frac{1+\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}=1+\frac{1}{2}P_1(\cos\alpha)+\frac{1}{3}P_2(\cos\alpha)+\frac{1}{4}P_3(\cos\alpha)+......$

(iv) $P'_{n+1}+P'_n=P_0+3P_1+5P_2+......+(2n+1)P_n$

(v) $\int_{-1}^{1}x^2P_{n+1}(x)P_{n-1}(x)\,dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

(vi) $\int_{-1}^{1}(1-x^2)[P'_n(x)]^2\,dx=\frac{2n(n+1)}{(2n+1)}$ (vii) $\int_{-1}^{1}xP_n(x)P_{n-1}(x)\,dx=\frac{2n}{4n^2-1}$

42. Prove that,

(i) $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$

(ii) $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

(iii) $\sqrt{\frac{\pi x}{2}}J_{3/2}(x)=\frac{\sin x}{x}-\cos x$

(iv) $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2\left[\frac{n}{x}J_n^2(x)-\frac{n+1}{x}J_{n+1}^2(x)\right]$

(x) $\cos(x.\sin\theta)=J_0+2J_2\cos2\theta+2J_4\cos4\theta+..................$

$\sin(x.\sin\theta)=2J_1\sin\theta+2J_3\sin3\theta+..........................$

(xi) $\frac{d}{dx}\left[x.J_n(x).J_{n+1}(x)\right]=x\left[J_n^2(x)-J_{n+1}^2(x)\right]$

43. Express the following polynomials as Hermite polynomials:

(i) $64x^4+8x^3-32x^2+40x+10$ [**Ans.** $4H_4(x)+H_3(x)+40H_2(x)+26H_1(x)+42H_0(x)$]

(ii) $8x^3+8x^2-6x+2$ [**Ans.** $H_3(x)+2H_2(x)+3H_1(x)+6H_0(x)$]

44. Find the value of: (i) $\int_{-\infty}^{\infty}e^{-x^2}H_{20}(x)H_{10}(x)dx$ (ii) $\int_{-\infty}^{\infty}e^{-x^2}\left[H_4(x)\right]^2dx$

45. Show that: (i) $L_3'(x)=-18+18x-3x^2$ (ii) $L_4''(x)=144-96x+12x^2$

Chapter 4

FOURIER SERIES

4.1 Periodic Function

If the value of function $f(x)$ repeats itself after equal interval of x, then $f(x)$ is said to be periodic function of x. Say

$$f(x) = f(x+T) = f(x+2T) = \ldots\ldots$$

then $f(x)$ is said to be periodic function of x with period T.

Example: (i) $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$: $T = 2\pi$

(ii) $\tan x$, $\cot x$: $T = \pi$

(iii) $\sin(5x + 2\pi) = \sin 5x = \sin 5\left(x + \frac{2\pi}{5}\right)$: $T = \frac{2\pi}{5}$

4.2 Fourier Series Expansion

A non-sinusoidal periodic function can be expressed by a sum of a set of sinusoidal oscillating functions or periodic functions i.e. mainly sine and cosine functions (or complex exponentials). This series is known as 'Fourier Series' expansion of the non-sinisoidal periodic function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

a_0, a_n, b_n are constants, known as Fourier co-efficients.

A periodic function $f(x)$ can be expanded into fourier series which consists of the following terms:

(i) a_0: a constant term or D. C. term

(ii) $a_1 \cos x$ and $b_1 \sin x$: Fundamental frequency term i.e term having lowest frequency

(iii) $a_2 \cos 2x$, $a_3 \cos 3x$....., $b_2 \sin 2x$, $b_3 \sin 3x$.... : Harmonics i.e terms having frequencies integer multiple of fundamental frequency term.

Note: When a function and its derivative are continuous, then the function can be expanded by powers of x by Taylor series. But for both continuous and discontinuous functions, fourier series expansion is possible under certain conditions.

Dirichlet's conditions

Suppose a function $f(x)$ is defined in the interval (a,b) and if

(i) it is single-valued in the interval (a,b)

(ii) it is bounded in the interval (a,b)

(iii) it has at most a finite number of maxima and minima in the interval (a,b)

(iv) it has only a finite number of discontinuity in the interval (a,b)

(v) $f(x+T) = f(x)$ where $[T = b - a]$, for all values of x outside $[a,b]$ then,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

4.2.2 Useful Integrals

(i) $\int_0^{2\pi} \sin nx dx = 0$ (ii) $\int_0^{2\pi} \cos nx dx = 0$ (iii) $\int_0^{2\pi} \sin^2 nx dx = \pi$

(iv) $\int_0^{2\pi} \cos^2 nx dx = \pi$ (v) $\int_0^{2\pi} \sin nx \sin mx\, dx = 0$ $[m \neq n]$

(vi) $\int_0^{2\pi} \cos nx \cos mx\, dx = 0$ $[m \neq n]$ (vii) $\int_0^{2\pi} \sin nx \cos mx\, dx = \int_0^{2\pi} \sin nx \cos nx\, dx = 0$

(viii) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$.

4.2.3 Evaluation of Fourier co-efficients:

CASE 1: Suppose a function f(x) is defined in the interval $(0,2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx\,;\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx\,;\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

CASE 2: Suppose a function f(x) is defined in the interval $(-\pi,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\,;\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,;\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

CASE 3: Suppose a function f(x) is defined in the interval $(0,2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx\,;\quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx\,;\quad b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$$

CASE 4: Suppose a function f(x) is defined in the interval $(-l, l)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx\,;\; a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx\,;\; b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx$$

CASE 5: Suppose a function $f(t)$ is defined in the interval $\left(0,\frac{2\pi}{\omega}\right)$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega t+\sum_{n=1}^{\infty}b_n\sin n\omega t$$

$$a_0=\frac{\omega}{\pi}\int_{0}^{2\pi/\omega}f(t)dt\,;\; a_n=\frac{\omega}{\pi}\int_{0}^{2\pi/\omega}f(t)\cos n\omega t\,dt\,;\; b_n=\frac{\omega}{\pi}\int_{0}^{2\pi/\omega}f(t)\sin n\omega t\,dt$$

CASE 6: Suppose a function $f(t)$ is defined in the interval $\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega t+\sum_{n=1}^{\infty}b_n\sin n\omega t$$

$$a_0=\frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega}f(t)dt\,;\; a_n=\frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega}f(t)\cos n\omega t\,dt\,;\; b_n=\frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega}f(t)\sin n\omega t\,dt$$

Nature of the fourier series for even function and odd function:

CASE 1: EVEN FUNCTION

A function is said to be even if $f(-x)=f(x)$

For a even function: $\int_{-\pi}^{\pi}f(x)dx=2\int_{0}^{\pi}f(x)dx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=0$$

This series will contain only cosine terms and the fourier series expansion can be written as

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

CASE 2: ODD FUNCTION

A function is said to be odd if $f(-x)=-f(x)$

For a odd function: $\int_{-\pi}^{\pi}f(x)dx=0$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\sin nx\,dx$$

This series will contain only sine terms and the fourier series expansion can be written as

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

Example 1: Find the fourier series representing, $f(x)=x, 0<x<2\pi$ and sketch its graph from $x=-4\pi$ to $x=4\pi$.

Soln: $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}xdx=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}=2\pi$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_0^{2\pi}x\cos nxdx$$

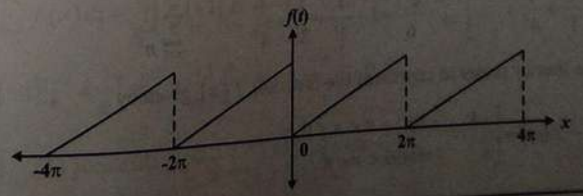
$$=\frac{1}{\pi}\left[x\frac{\sin nx}{n}-1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2}-\frac{1}{n^2}\right]=\frac{1}{n^2\pi}(1-1)=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_0^{2\pi}x\sin nxdx$$

$$=\frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right]=-\frac{2}{\pi}$$

The required fourier series expansion of the function:

$$x=\pi-2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+......\right]$$



Example 2: Given that $f(x)=x+x^2$ for $-\pi<x<\pi$, find the Fourier expression of $f(x)$. Deduce that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots$$

Soln: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx$

$$=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\int_{-\pi}^{\pi}(x+x^2)\cos nxdx$$

$$=\frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n}-(2x+1)\frac{(-\cos nx)}{n^2}+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2}-(-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right]=\frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right]=\frac{4(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)x\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nxdx$$

$$=\frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(2x+1)\left(\frac{-\sin nx}{n^2}\right)+2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n}+2\frac{\cos n\pi}{n^3}+(-\pi+\pi^2)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}\right]$$

$$=\frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right]=-\frac{2}{n}(-1)^n$$

The required fourier series series expansion of the function:

$$x+x^2=\frac{\pi^3}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\ldots.\right]-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\ldots.\right]$$

Putting $x=\pi$ we get, $\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.\right]$

Putting $x=-\pi$ we get, $-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right]$

$$\Rightarrow\frac{4\pi^2}{3}=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots..\right]\Rightarrow\ \frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots..=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Example 3: Find the fourier series to represent the function $f(x)$ given by

$$f(x)=\begin{cases}-k & \text{for } -\pi<x<0\\ k & \text{for } 0<x<\pi\end{cases}$$

hence show that $1-\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots=\frac{\pi}{4}$

Soln: Given function is odd in nature, so $a_0=0$ and $a_n=0$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-k\sin nxdx+\int_{0}^{\pi}k\sin nxdx\right]$$

$$=\frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0}-\left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}k\left[\frac{1}{n}-\frac{(-1)^n}{n}-\frac{(-1)^n}{n}+\frac{1}{n}\right]=\frac{1}{\pi}k\left[\frac{2}{n}-\frac{2(-1)^n}{n}\right]$$

If n is even $b_n=0$

If n is odd $b_n=\frac{4k}{n\pi}$

Thus required fourier sine series is

$$f(x)=\frac{4k}{\pi}\sin x+\frac{4k}{3\pi}\sin 3x+\frac{4k}{5\pi}\sin 5x+\ldots..$$

$$\Rightarrow\qquad f(x)=\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots.\right]$$

Putting $x=\frac{\pi}{2}$, we get

$$k=\frac{4k}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\ldots..\right]$$

$$\Rightarrow 1=\frac{4}{\pi}\left[1+\frac{1}{3}(-1)+\frac{1}{5}(1)+\frac{1}{7}(-1)+\ldots\ldots\right]=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.\right]\Rightarrow\ \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots$$

Example 4: Find the fourier series of the function:

$$f(x)=\begin{cases}-1 & \text{for } -\pi<x<-\frac{\pi}{2}\\ 0 & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ +1 & \text{for } \frac{\pi}{2}<x<\pi\end{cases}$$

Soln: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)dx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}0\,dx+\frac{1}{\pi}\int_{\pi/2}^{\pi}1\,dx$

$$=\frac{1}{\pi}[-x]_{-\pi}^{-\pi/2}+\frac{1}{\pi}[x]_{\pi/2}^{\pi}=\frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\pi-\frac{\pi}{2}\right]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\quad=\frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)\cos nxdx+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nxdx+\frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nxdx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} \qquad = -\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] + \frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1)\sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$b_1 = \frac{2}{\pi}$, $b_2 = -\frac{2}{\pi}$, $b_3 = \frac{2}{3\pi}$, therefore the fourier series expansion becomes

$$f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \ldots..\right]$$

Example 5: Find the fourier seires of the function defined as

$$f(x) = \begin{cases} x+\pi, & \text{for } 0 \le x \le \pi \\ -x-\pi, & \text{for } -\pi \le x < 0 \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$

Soln: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)dx = \frac{1}{\pi}\left(-\frac{x^2}{2} - \pi x\right)_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x^2}{2} + \pi x\right)_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} - \pi^2\right) + \frac{1}{\pi}\left(\frac{\pi^2}{2} + \pi^2\right) = \pi\left(\frac{1}{2} - 1\right) + \pi\left(\frac{1}{2} + 1\right) = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nx dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n} - (-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n} - (-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$= \frac{-4}{n^2\pi}$, if n is odd

$= 0$, if n is even

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nx dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(\frac{-\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right] = \frac{1}{n}\left[(1) - 2(-1)^n\right] = \frac{2}{n}\left[1 - (-1)^n\right]$$

$= \frac{4}{n}$, if n is odd

$= 0$, if n is even

The fourier series expansion of the function is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots..$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots..\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots..\right)$$

Example 6: Find the Fourier series expansion of the periodic function of period 2π

$$f(x) = x^2, \quad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots...$

Soln: $f(x) = x^2, \quad -\pi \le x \le \pi$

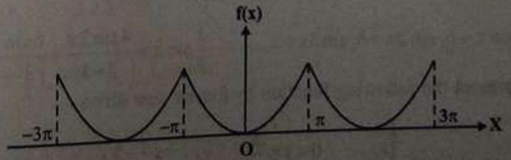
This is an even function, $\quad \therefore b_n = 0$

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_{0}^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_{0}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2\sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series expansion: $x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots.\right]$

On putting $x = 0$, we have $0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots..\right]$

$$\Rightarrow \quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\dots=\frac{\pi^2}{12}$$

Example 7: Find the Fourier Series expansion for the function

$$f(x)=x\cos x,-\pi<x<\pi$$

Soln: Since $x\cos x$ is an odd function therefore, $a_0=a_n=0$

$$b_n=\frac{2}{\pi}\int_0^\pi x\sin(n+1)xdx+\frac{1}{\pi}\int_0^\pi x\sin(n-1)xdx$$

$$=\frac{1}{\pi}\int_0^\pi x\sin(n+1)xdx+\frac{1}{\pi}\int_0^\pi x\sin(n-1)dx$$

$$=\frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right)+\frac{\sin(n+1)x}{(n+1)^2}\right]_0^\pi+\frac{1}{\pi}\left[-x\frac{\cos(n-1)x}{n-1}+\frac{\sin(n-1)x}{(n-1)^2}\right]_0^\pi$$

$$=\frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{(n+1)}-\frac{\cos(n-1)x}{(n-1)}\right\}+1.\left\{\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$=\frac{1}{\pi}\left[\pi.\left\{-\frac{\cos(n+1)\pi}{(n+1)}-\frac{\cos(n-1)\pi}{(n-1)}\right\}\right]\Rightarrow b_n=\left\{-\frac{(-1)^{n+1}}{n+1}-\frac{(-1)^{n-1}}{n-1}\right\},\ n\neq 1$$

$$b_n=-(-1)^{n+1}\left[\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$=-\left\{\frac{1}{(n+1)}+\frac{1}{(n-1)}\right\}=\frac{-2n}{n^2-1} \qquad \text{If n is odd, } n\neq 1$$

But $b_n=\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}=\frac{2n}{n^2-1}$, If n is even, $n\neq 1$

If n = 1, then $b_1=\frac{2}{\pi}\int_0^\pi x\cos x.\sin xdx=\frac{1}{\pi}\int_0^\pi x.\sin 2xdx$

$$=\frac{1}{\pi}\left[x.\left(-\frac{\cos 2x}{2}\right)-1\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi=\frac{1}{\pi}\left[\pi\left(-\frac{1}{2}\right)\right]=-\frac{1}{2}$$

$$\therefore \quad x\cos x=b_1\sin x+b_2\sin 2x+b_3\sin 3x+\dots\dots=-\frac{1}{2}\sin x+\frac{4\sin 2x}{2-1}-\frac{6\sin 3x}{3^2-1}+\dots..$$

Example 8: Represent the following function by fourier sine series

$$f(t)=\begin{cases} t, & 0<t\le\frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2}<t\le\pi \end{cases}$$

Soln: $b_n=\frac{2}{\pi}\int_0^\pi f(t)\sin ntdt \quad =\frac{2}{\pi}\int_0^{\pi/2} t\sin ntdt+\frac{2}{\pi}\int_{\pi/2}^\pi \frac{\pi}{2}\sin ntdt$

$$=\frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right)-(1)\left(-\frac{\sin nt}{n^2}\right)\right]_0^{\pi/2}+\frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^\pi$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]+\left[-\frac{\cos n\pi}{n}+\frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1=\frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right]+\left[-\cos\pi+\cos\frac{\pi}{2}\right]=\frac{2}{\pi}[0+1]+[1]=\frac{2}{\pi}+1$$

$$b_2=\frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2}+\frac{\sin\pi}{2}\right]+\left[-\frac{\cos 2\pi}{2}+\frac{\cos\pi}{2}\right]=\frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2}+0\right]+\left[-\frac{1}{2}-\frac{1}{2}\right]$$

$$=\frac{2}{\pi}\left[\frac{\pi}{4}\right]-1=\frac{1}{2}-1=-\frac{1}{2}$$

$$b_3=\frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3}+\frac{\sin\frac{3\pi}{2}}{3^2}\right]+\left[-\frac{\cos 3\pi}{3}+\frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2}(0)-\frac{1}{9}\right]+\left[\frac{1}{3}+0\right]=-\frac{2}{9\pi}+\frac{1}{3}$$

$$f(t)=\left(\frac{2}{\pi}+1\right)\sin t-\frac{1}{2}\sin 2t+\left(-\frac{2}{9\pi}+\frac{1}{3}\right)\sin 3t+\dots..$$

Example 9: A periodic function of period 4 is defined as

$$f(x)=|x|.\ -2<x<2$$

Find its Fourier series expansion.

Soln: $f(x)=|x|.\ -2<x<2$

$$\Rightarrow \quad f(x)=\begin{cases} x, & 0<x<2 \\ -x, & -2<x<0 \end{cases}$$

$$a_0=\frac{1}{c}\int_{-c}^c f(x)dx=\frac{1}{2}\int_0^2 xdx+\frac{1}{2}\int_{-2}^0(-x)dx$$

$$=\frac{1}{2}\left[\frac{x^2}{2}\right]_0^2+\frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^0=\frac{1}{4}(4-0)+\frac{1}{4}(0+4)=2$$

$$a_n=\frac{1}{c}\int_{-c}^c f(x)\cos\frac{n\pi x}{c}dx=\frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx+\frac{1}{2}\int_{-2}^0(-x)\cos\frac{n\pi x}{2}dx$$

$$=\frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^0$$

$$=\frac{1}{2}\left[0+\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}\right]+\frac{1}{2}\left[0-\frac{4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right]$$

$$=\frac{1}{2}\frac{4}{n^2\pi^2}\left[(-1)^n-1-1+(-1)^n\right]=\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]$$

$=-\frac{8}{n^2\pi^2}$ (If n is odd)

$=0$ (if n is even)

$b_n=0$ as $f(x)$ is even function

Therefore, the fourier series exapansion will be

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{2}+a_2\cos\frac{2\pi x}{2}+\ldots+b_1\sin\frac{\pi x}{2}+b_2\sin\frac{2\pi x}{2}+\ldots..$$

$$f(x)=1-\frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2}+\frac{\cos\frac{3\pi x}{2}}{3^2}+\frac{\cos\frac{5\pi x}{2}}{5^2}+\ldots..\right]$$

4.3 Half Range Series

Let the function is defined in the interval (0, π) and it is insignificant whatever be the form of function outside (0, π).

(1) To get cosine half range series, we take f(x) is an even function in the interval (–π, π)

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx\ ;\ a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx\ ;\ b_n=0\ ;$$

(2) To get sine half range series, we take f(x) is an odd function in the interval (–π, π)

$$a_0=0;\ a_n=0;\ b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx\ ;$$

4.4 Perseval's Formula

Full range Fourier series: $\int_{-l}^{l}[f(x)]^2dx = l\left\{\frac{1}{2}a_0^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\right\}$

Half range cosine series : $\int_0^{l}[f(x)]^2dx=\frac{l}{2}\left[\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2\right]$

Half range sine series: $\int_0^{l}[f(x)]^2dx=\frac{l}{2}\left[\sum_{n=1}^{\infty}b_n^2\right]$

Example 10: By using the half range sine series for $f(x)=1$ in $0<x<\pi$, show that

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots.$$

Soln: $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nxdx$

$$=\frac{2}{\pi}\int_0^{\pi}(1)\sin nxdx=\frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi}=\frac{-2}{n\pi}[\cos n\pi-1]=\frac{-2}{n\pi}\left[(-1)^n-1\right]$$

$=\frac{4}{n\pi}$ if n is odd

$=0$ if n is even

Then the sine series is

$$1=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x+\frac{4}{7\pi}\sin 7x+\ldots..$$

Using Perseval's formula, we get

$$\int_0^{l}\left[f(x)\right]^2dx=\frac{l}{2}\left[b_1^2+b_2^2+b_3^2+b_4^2+b_5^2+\ldots..\right]$$

$$\Rightarrow\int_0^{\pi}(1)^2dx=\frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2+\left(\frac{4}{3\pi}\right)^2+\left(\frac{4}{5\pi}\right)^2+\left(\frac{4}{7\pi}\right)^2+\ldots\right]$$

$$\Rightarrow[x]_0^{\pi}=\left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots...\right]\Rightarrow\pi=\frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots.\right]$$

$$\Rightarrow\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots..$$

Example 11: If $f(x)=\begin{cases}\pi x & 0<x<1\\ \pi(2-x), & 1<x<2\end{cases}$

Using half range cosine series, show that

$$\frac{\pi^4}{96}=1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots...$$

Soln: $a_0=\frac{2}{l}\int_0^{l}f(x)\,dx=\frac{2}{2}\left[\int_0^{1}\pi xdx+\int_1^{2}\pi(2-x)dx\right]$

$$=\pi\left(\frac{x^2}{2}\right)_0^1+\pi\left(2x-\frac{x^2}{2}\right)_1^2=\frac{\pi}{2}+\pi\left[(4-2)-\left(2-\frac{1}{2}\right)\right]=\pi$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{2}\left[\int_0^1 \pi x\cos\frac{n\pi x}{2}dx+\int_1^2 \pi(2-x)\cos\frac{n\pi x}{2}dx\right]$$

$$=\pi\left[x\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}-\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1+\pi\left[(2-x)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}-(-1)\left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]\right]_1^2$$

$$=\pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2}+\frac{4}{n2}\pi^2\cos\frac{n\pi}{2}-\frac{4}{n^2\pi^2}\right]+\pi\left[0-\frac{4}{n^2\pi^2}\cos n\pi-\frac{2}{n\pi}\sin\frac{n\pi}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$=\pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{4}{n^2\pi^2}-\frac{4}{n^2\pi^2}\cos n\pi\right]=\frac{4}{n^2\pi^2}\left[2\cos\frac{n\pi}{2}-1-\cos n\pi\right]$$

$$a_1=0, a_2=-\frac{4}{\pi},\ a_3=0,\ a_4=0, a_5=0,\ a_6=-\frac{4}{9\pi}.....$$

According to Perseval's formula:

$$\int_0^l[f(x)]^2dx=\frac{l}{2}\left[\frac{a_0^2}{2}+a_1^2+a_2^2+a_3^2+.......\right]$$

$$\Rightarrow\int_0^1(\pi x)^2dx+\int_1^2\pi^2(2-x)^2dx=\frac{2}{2}\left[\frac{\pi^2}{2}+\frac{16}{\pi^2}+\frac{16}{81\pi^2}+.....\right]$$

$$\Rightarrow\pi^2\left[\frac{x^3}{3}\right]_0^1-\pi^2\left[\frac{(2-x)^3}{3}\right]_1^2=\frac{\pi^2}{2}+\frac{16}{\pi^2}+\frac{16}{81\pi^2}+......\Rightarrow\frac{\pi^2}{3}-\pi^2\left(0-\frac{1}{3}\right)=\frac{\pi^2}{2}+\frac{16}{\pi^2}\left[1+\frac{1}{81}+......\right]$$

$$\Rightarrow\frac{2\pi^2}{3}-\frac{\pi^2}{2}=\frac{16}{\pi^2}\left[1+\frac{1}{3^4}+\frac{1}{5^4}+.....\right]\Rightarrow\frac{\pi^2}{6}=\frac{16}{\pi^2}\left[1+\frac{1}{3^4}+\frac{1}{5^4}+.....\right]$$

$$\Rightarrow\frac{\pi^4}{96}=1+\frac{1}{3^4}+\frac{1}{5^4}+......$$

4.5 Fourier series in complex form

Suppose a function f(x) is defined in the interval $(0,2l)$

$$f(x)=c_0+\sum_{n=1}^{\infty}c_n e^{i\frac{n\pi x}{l}}+\sum_{n=1}^{\infty}c_{-n}e^{-i\frac{n\pi x}{l}}$$

$$c_0=\frac{1}{2l}\int_0^{2l}f(x)dx;\quad c_n=\frac{1}{2l}\int_0^{2l}f(x)e^{-i\frac{n\pi x}{l}}dx;\quad c_{-n}=\frac{1}{2l}\int_0^{2l}f(x)e^{i\frac{n\pi x}{l}}dx$$

Example 12: Obtain the complex form of the Fourier series of the function

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ 1, & 0\le x\le\pi\end{cases}$$

Soln: $c_0=\frac{1}{2\pi}\int_0^{\pi}dx=\frac{1}{2}$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0.e^{-inx}dx+\int_0^{\pi}1.e^{-inx}dx\right]=\frac{1}{2\pi}\int_0^{\pi}e^{-inx}dx=\frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$=-\frac{1}{2n\pi i}\left[e^{-inx}-1\right]=-\frac{1}{2n\pi i}\left[\cos n\pi-i\sin n\pi-1\right]=-\frac{1}{2n\pi i}\left[(-1)^n-0-1\right]$$

$$c_n=\begin{cases}\frac{1}{in\pi}, & n\text{ is odd}\\ 0, & \text{n is even}\end{cases}$$

Similarly, $c_{-n}=\begin{cases}-\frac{1}{in\pi}, & n\text{ is odd}\\ 0, & \text{n is even}\end{cases}$

$$f(x)=\frac{1}{2}+\frac{1}{i\pi}\left[\frac{e^{ix}}{1}+\frac{e^{3ix}}{3}+\frac{e^{5ix}}{5}+.....\right]+\frac{1}{i\pi}\left[\frac{e^{-ix}}{-1}+\frac{e^{-3ix}}{-3}+\frac{e^{-5ix}}{-5}+.......\right]$$

$$=\frac{1}{2}-\frac{1}{i\pi}\left[\left(e^{ix}-e^{-ix}\right)+\frac{1}{3}\left(e^{3ix}-e^{-3ix}\right)+\frac{1}{5}\left(e^{5ix}-e^{-5ix}\right)+......\right]$$

$$=\frac{1}{2}-\frac{2}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+.......\right]$$

GATE Previous Years Questions with Solutions

1. $f(x)$ is symmetric periodic function of x i.e. $f(-x)=f(x)$. Then, in general, the Fourier series of the function $f(x)$ will be of the form **[GATE-2013]**

(a) $f(x)=\sum_{n=1}^{\infty}\left(a_n\cos(nkx)+b_n\sin(nkx)\right)$

(b) $f(x)=a_0+\left(a_n\cos(nkx)\right)$

(c) $f(x)=\sum_{n=1}^{\infty}\left(b_n\sin(nkx)\right)$

(d) $f(x)=a_0+\sum_{n=1}^{\infty}\left(b_n\sin(nkx)\right)$

Soln. For a symmetric periodic function i.e., $f(-x)=f(x)$.

The Fourier series contains constant term and cosine terms.

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(nkx)$$

Correct option is (b)

IIT-JAM Previous Year Questions with Solutions

1. A periodic function can be expressed in a Fourier series of the form, $f(x)=\sum_{n=0}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)$. The functions $f_1(x)=\cos^2 x$ and $f_2(x)=\sin^2 x$ are expanded in their respective Fourier series. If the coefficients for the first series are $a_n^{(1)}$ and $b_n^{(1)}$, and the coefficients for the second series are $a_n^{(2)}$ and $b_n^{(2)}$, respectively, then which of the following is correct?

(a) $a_2^{(1)}=\frac{1}{2}$ and $b_2^{(2)}=\frac{-1}{2}$ (b) $b_2^{(1)}=\frac{1}{2}$ and $a_2^{(2)}=\frac{-1}{2}$

(c) $a_2^{(1)}=\frac{1}{2}$ and $a_2^{(2)}=\frac{-1}{2}$ (d) $b_2^{(1)}=\frac{1}{2}$ and $b_2^{(2)}=\frac{-1}{2}$

Soln. Both $f_1(x)=\cos^2 x$ and $f_2(x)=\sin^2 x$ are even functions.

So, $b_n^{(1)}=b_n^{(2)}=0$ and $a_n^{(1)},a_n^{(2)}\neq 0$

Correct option is (c)

2. f(x) is a periodic function of x with a period of 2π. In the interval $-\pi<x<\pi$, f(x) is given by

$$f(x)=\begin{cases}0, & -\pi<x<0\\ \sin x, & 0<x<\pi\end{cases}$$

In the expansion of f(x) as a Fourier series of sine and cosine functions, the coefficients of cos (2x) is:

(a) $\frac{2}{3\pi}$ (b) $\frac{1}{\pi}$ (c) 0 (d) $-\frac{2}{3\pi}$

Soln. Coefficient of $\cos 2x=a_2=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos 2x\,dx$

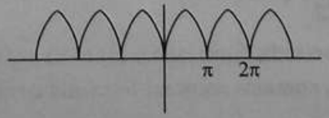
$$=\frac{1}{\pi}\int_0^{\pi}\sin x.\cos 2x\,dx=\frac{1}{2\pi}\int_0^{\pi}[\sin 3x-\sin x]dx=-\frac{2}{3\pi}$$

Correct answer is (d)

3. In the Fourier series of the periodic function (shown in the figure)

$f(x)=|\sin x|$

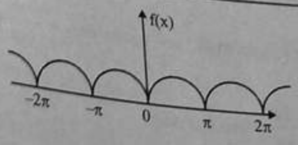
$$=\sum_{n=0}^{\infty}[\alpha_n\cos nx+\beta_n\sin nx]$$



Which of the following coefficients are nonzero?

(a) α_n for odd n (b) α_n for even n (c) β_n for odd n (d) β_n for even n

Soln. $f(x)=|\sin x|$ for $0<x<\pi$
$=\sin x$ for $0<x<\pi$
$=-\sin x$ for $-\pi<x<0$



Since, it is an even function $\beta_n=0$ for all n.

$$\alpha_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x.\cos nx\,dx$$

$$=\frac{2}{2\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]dx=\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)}+\frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)}+\frac{1}{(n+1)}+\frac{(-1)^{n-1}}{(n-1)}-\frac{1}{(n-1)}\right]$$

$=0$ for odd 'n'

$=-\frac{4}{\pi(n^2-1)}$ for even 'n'

Correct option is (b)

4. Consider a periodic function $f(x)$, with periodicity 2π

$$f(x)=\begin{cases}c & 0\le x<\pi\\ 0 & \pi\le x<2\pi\end{cases}$$

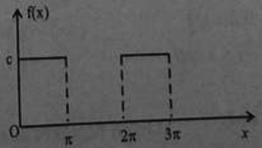
where 'c' is a constant.

(a) Expand $f(x)$ in a fourier series.

(b) From the result obtained in (a). Show that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\ldots\ldots\ldots$$

Soln. $f(x)=\begin{cases}c & 0\le x<\pi\\ 0 & \pi\le x\le 2\pi\end{cases}$



(a) Fourier series expansion of $f(x)$:

$$f(x)=\frac{a_0}{2}+\sum_n a_n\cos nx+\sum_n b_n\cos nx$$

$$a_0=\frac{1}{\pi}\left[\int_0^{2\pi}f(x)dx\right]=\frac{1}{\pi}\left[\int_0^{\pi}c.dx+\int_{\pi}^{2\pi}0.dx\right]=c$$

$$a_n=\frac{1}{\pi}\left[\int_0^{2\pi}f(x)\cos nx\,dx\right]=\frac{1}{\pi}\int_0^{\pi}c.\cos nx\,dx=\frac{c}{\pi}\frac{\sin nx}{n}\Big|_0^{\pi}=0$$

$$b_n = \frac{1}{\pi}\left[\int_0^{2\pi} f(x)\sin nx dx\right] = \frac{1}{\pi}\int_0^{\pi} c.\sin nx dx = -\frac{c}{n\pi}\cos nx\Big|_0^{\pi} = -\frac{c}{n\pi}\left[(-1)^n - 1\right]$$

So, $b_n = \frac{2c}{n\pi}$ when n is odd

$= 0$ when n is even

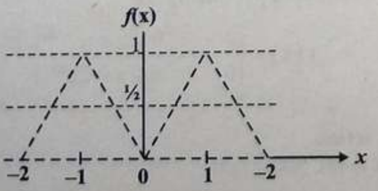
Therefore, $f(x) = \frac{c}{2} + \frac{2c}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\ldots\right]$

(b) Putting at $x = \frac{\pi}{2}, f(x) = c$

$\Rightarrow \quad c = \frac{c}{2} + \frac{2c}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\right]$

$\Rightarrow \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots = \frac{\pi}{4}$

5. The Fourier series for an arbitrary periodic function with period $2L$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$. For the particular periodic function shown in the figure, the value of a_0 is



(a) 0 (b) 0.5 (c) 1 (d) 2

Soln. Functional form :

$f(x) = x \quad 0 \le x \le 1$

$= -x \quad -1 \le x \le 0$

Then, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$ [Given function has time period T = 2]

$\Rightarrow \quad a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = \left[\int_{-1}^{0} -xdx + \int_0^1 xdx\right]$

$$= \left[\left\{\frac{-x^2}{2}\right\}_{-1}^{0} + \left\{\frac{x^2}{2}\right\}_0^1\right] = \left[\frac{1}{2} + \frac{1}{2}\right] = 1$$

Correct option is (c)

PRACTICE SET

1. Which of the following is NOT a periodic function of θ?

(a) $\sin\theta + \pi\cos\theta$

(b) $\sin(\pi\theta) + \cos(\pi\theta)$

(c) $\sin\theta + \cos(\pi\theta)$

(d) $\sin\theta + \cos(\theta + \pi)$

2. In the fourier series expansion of $y = (x^2 - 1)^{1/2}$ $(-\pi \le x \le \pi)$ the coefficient a_1, b_1 will be

(a) $\frac{1}{\pi}, 1$ (b) $1, -\frac{1}{\pi}$ (c) $2, \frac{\pi^2}{2}$ (d) Unable to find

3. In the Fourier series expansion of $y = \frac{1}{x^2(x-2)}$ $(-2 \le x \le 2)$ the co-efficient a_0 will be

(a) Unable to find (b) 0 (c) $\frac{\pi^2}{3}$ (d) $\frac{2}{3}$

4. Suppose f(x) is defined as following:

$f(x) = 1; \quad -1 \le x < 0$

$= -2; \quad 0 \le x < 1$

then a_0, a_3, b_3 in the fourier series exapansion of f(x) are

(a) $\left(0, 0, -\frac{2}{\pi}\right)$ (b) $\left(-1, 0, -\frac{2}{\pi}\right)$ (c) $\left(1, 0, \frac{2}{\pi}\right)$ (d) $\left(-1, 0, \frac{2}{\pi}\right)$

5. If $f(x) = -k; \quad -\pi < x < 0$

$= k; \quad 0 < x < \pi$

then co-efficient b_2, b_3 in the fourier series exapnasion will be

(a) $\frac{4k}{2\pi}, \frac{4k}{3\pi}$ (b) $-\frac{4k}{2\pi}, -\frac{4k}{3\pi}$ (c) $0, \frac{4k}{3\pi}$ (d) $\frac{4k}{2\pi}, 0$

6. f(x) is periodic function with $-\pi < x < \pi$ and is given by,

$f(x) = 0; \quad -\pi < x < 0$

$= 2\pi; \quad 0 < x < \pi$

In the fourier series expansion, the coefficient of $\cos 2x$ is:

(a) $\frac{2}{3\pi}$ (b) $\frac{1}{\pi}$ (c) 0 (d) $-\frac{2}{3\pi}$

7. If $f(x) = 0; \quad -\pi < x \le 0$

$= \frac{\pi x}{4}; \quad 0 < x < \pi$

then the co-efficient a_n will be

(a) $\frac{(-1)^n}{4n^2}$ (b) $\frac{(-1)^n - 1}{4n^2}$ (c) $\frac{(-1)^n - 1}{4n}$ (d) $\frac{(-1)^n}{4n}$

8. In the fourier expansion for $f(x) = x\cos x\,(-\pi < x < \pi)$. Which of the following statement is true?

(a) $b_n = 0$ for all n

(b) $b_n = \frac{2n}{n^2-1}$ for all even n

(c) $b_n = -\frac{2n}{n^2-1}$ for all even n

(d) $b_n = -\frac{2n}{n^2-1}$

9. In the fourier series expansion of $f(x) = |\sin x|$, which of the following co-efficients are non-zero? $(-\pi \le x \le \pi)$

(a) a_n for odd n

(b) a_n for even n

(c) b_n for odd n

(d) b_n for even n.

10. If $f(x) = x - x^2$ from $x = -\pi$ to π is expanded into

$$f(x) = -\frac{\pi^2}{3} + \sum \frac{4}{n^2}(-1)^{n+1}\cos nx + \sum \frac{2}{n}(-1)^{n+1}\sin nx$$

then $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ is equal to

(a) $\frac{\pi^2}{12}$

(b) $\frac{-\pi^2}{12}$

(c) $\frac{\pi^2}{4}$

(d) $\frac{-\pi^2}{4}$

11. The function $f(x) = x^2, 0 < x < 2$ has a Fourier series representation

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

At $x = 0$ and 2π, the series converges to $2\pi^2$. Then the value of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is:

(a) $2\pi^2$

(b) π^2

(c) $\frac{\pi^2}{3}$

(d) $\frac{\pi^2}{6}$

12. If f(x) be an even, period function of x which is continuous. Then

(a) It can be represented by a cosine series.

(b) It can be represented by a sine series

(c) It cannot be represented by a Fourier series

(d) It needs both sine and cosine terms in its Fourier expansion

13. $f_1(x) = \sin^2 x$ and $f_2(x) = \cos^2 x$ are expanded in their corresponding fourier series. Which of the following is correct?

(a) $a_2^{(1)} = \frac{1}{2}, b_2^{(2)} = -\frac{1}{2}$

(b) $b_2^{(1)} = \frac{1}{2}, a_2^{(2)} = -\frac{1}{2}$

(c) $a_2^{(1)} = \frac{1}{2}, a_2^{(2)} = -\frac{1}{2}$

(d) $b_2^{(1)} = \frac{1}{2}, b_2^{(2)} = -\frac{1}{2}$

---

14. Consider the Fourier series corresponding to the function

$$f(x) = \begin{cases} 3, & 0 < x < 5 \\ -3, & -5 < x < 0 \end{cases}$$

Then the fourier series contains.

(a) Cosine terms only

(b) Both cosine and sine terms

(c) Sine terms only

(d) Sine terms and non-zero constant terms.

15. The Fourier series expansion of the function $f(x) = x(\pi - x), 0 \le x \le t$, is:

$$\frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right).$$

The value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is deduced as

(a) $\frac{\pi^2}{6}$

(b) $\frac{2\pi^2}{6}$

(c) $-\frac{\pi^2}{6}$

(d) Zero

16. The Fourier coefficients a_n in the Fourier series expansion.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \text{ for } f(x) = \begin{cases} -1, & -\pi \le x \le 0 \\ +1, & 0 \le x \le \pi \end{cases}, \text{ are}$$

(a) $\frac{2}{n\pi}\left[1 - (-1)^n\right]$

(b) $\frac{4}{n\pi}$

(c) $\frac{2}{n\pi}\left[1 + (-1)^n\right]$

(d) 0

17. A periodic function $f(x) = x$ for $-\pi < x < \pi$ has fourier series representation

$$f(x) = \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)(-1)^n \sin nx$$

Using this, one fields the sum $\sum_{n=1}^{\infty} n^{-2}$ to be

(a) $2\,\ell n\,2$

(b) $\frac{\pi^2}{3}$

(c) $\frac{\pi^2}{6}$

(d) $\pi\,\ell n\,2$

18. Find the fourier series expansion for

(i) Rectangular Wave: $f(x) = \left\{\begin{matrix} -k & \text{for } -\pi < x < 0 \\ k & \text{for } \ 0 < x < \pi \end{matrix}\right\}$

(ii) Half Wave Rectifier: $f(x) = \left\{\begin{matrix} 0 & \text{for } -\frac{\pi}{\omega} < t < 0 \\ E\sin\omega t & \text{for } \ 0 < t < \frac{\pi}{\omega} \end{matrix}\right\}$

(iii) Saw Tooth Wave: $f(x) = x + \pi$ for $-\pi < x < \pi$

Ans. (i) $\frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\dots\dots\right]$

(ii) $\frac{E}{\pi} + \frac{E}{2}\sin \omega t - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + \dots\dots\dots\dots\right]$

(iii) $\pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\dots\dots\right]$

19. Obtain the complex form of the fourier series of the expansion

$$\begin{aligned} f(x) &= 0; \quad -\pi \le x \le 0 \\ &= 1; \quad 0 \le x \le \pi \end{aligned}$$

Ans. $\frac{1}{2} + \frac{1}{i\pi}\left[\frac{e^{ix}}{1} + \frac{e^{i3x}}{3} + \frac{e^{i5x}}{5} + \dots\dots\dots\right] + \frac{1}{i\pi}\left[-\frac{e^{-ix}}{1} - \frac{e^{-3ix}}{3} - \frac{e^{-5ix}}{5} + \dots\dots\right]$

20. Obtain the (i) Half range Cosine series, (ii) Half range Sine series of the following function:

$$f(x) = \begin{Bmatrix} \frac{2k}{L}x & \text{for } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{for } \frac{L}{2} < x < L \end{Bmatrix}$$

Ans. (i) $\frac{k}{2} - \frac{16k}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \dots\dots\dots\dots\right]$

(ii) $\frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L} + \dots\dots\dots\dots\right]$

ANSWER KEY

1. (c)	2. (d)	3. (a)	4. (b)	5. (c)	6. (c)	7. (b)
8. (b)	9. (b)	10. (a)	11. (d)	12. (a)	13. (c)	14. (c)
15. (a)	16. (d)	17. (c)				

Chapter 5

COMPLEX ANALYSIS

5.1 Basic Review of Complex Numbers

Various representation of Complex number :

A complex number (z) is represented as $z = x + iy$, where x is the real part and y is imaginary part. The conjugate of complex number z is represented by $\overline{z} = x - iy$

The modulus of complex number $z = |z| = r = \sqrt{x^2 + y^2}$ and argument of $z = Arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

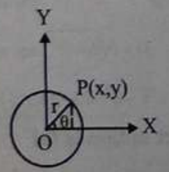
The complex number (z) in polar form (r, θ) can also be written as

$$z = x + iy = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$$

where $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = r^2 = |z|^2$



This implies that $|z| = r$ represents a circle centred at the origin and having radius r.

Properties of modulus of z:

If z_1 and z_2 are two complex numbers, then

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ (ii) $|z_1 - z_2| \ge |z_1| - |z_2|$

(iii) $|z_1 z_2| = |z_1||z_2|$ (iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Properties of argument of z:

(i) $Arg(z_1.z_2.z_3.\dots.z_n) = Arg.(z_1) + Arg.(z_2) + Arg.(z_3) + \dots\dots\dots + Arg(z_n)$

(ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$

Some important Relations:

(i) $e^{i\theta} = \cos\theta + i\sin\theta$, (ii) $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, (iii) $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$

(iv) $\cos h\theta = \frac{(e^{\theta} + e^{-\theta})}{2}$, (v) $\sin h\theta = \frac{(e^{\theta} - e^{-\theta})}{2}$

De-Moiver's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Complex cube roots of unity:

$x^3 = 1 \Rightarrow x = 1, \omega = \dfrac{-1+i\sqrt{3}}{2}, \omega^2 = \dfrac{-1-i\sqrt{3}}{2}$ such that $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$

Example 1: Multiplying a complex number z by $1 + i$ rotates the radius vector of z by an angle of

(a) 90° clockwise (b) 45° anticlockwise

(c) 45° clockwise (d) 90° anticlockwise

Soln: $z = r\,e^{i\theta} \Rightarrow (1+i)z = \left(\sqrt{2}\,e^{i\pi/4}\right) r\,e^{i\theta} = \sqrt{2}\,r\,e^{i(\theta+\pi/4)}$

Multiplying $(1 + i)$ with z rotates the radius vector anticlockwise by 45° and increases the modulus by a factor of $\sqrt{2}$.

Correct option is (a)

Example 2: If $z = (\lambda+3) + i\sqrt{5-\lambda^2}$ (λ is a real parameter and $i = \sqrt{-1}$), then the locus of z will be

(a) circle (b) ellipse (c) parabola (d) hyperbola

Soln: $z = (\lambda+3) + i\sqrt{5-\lambda^2} \Rightarrow$ real part $x = \lambda + 3$ and imaginary part $y = \sqrt{5-\lambda^2}$

$\Rightarrow y^2 = 5 - \lambda^2 = 5 - (x-3)^2 \Rightarrow (x-3)^2 + y^2 = 5$ (Equation of circle)

Correct option is (a)

5.2 Function of A complex Variable

Basic Representation:

$$f(z) = u(x,y) + iv(x,y)$$

Example: $f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2 - y^2)^2}_{u(x,y)\text{ is real part}} + \underbrace{i2xy}_{v(x,y)\text{ is imaginary part}}$

Existance of $\lim\limits_{z\to z_0} f(z)$:

The limit will exists only if the limiting value is independent of the path along which z approaches z_0

Example 3: Find whether the limit $\lim\limits_{z\to 0}\dfrac{z}{|z|}$ exist or not.

Soln: $z \to 0$ means $x \to 0$ & $y \to 0$

For $z = 0$, we have to choose a path passing through a origin.

Therefore, we have choosen a straight line passing through the origin i.e. $y = mx$

$$\lim_{z\to 0}\frac{z}{|z|} = \lim_{\substack{z\to 0\\ y\to 0}}\frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\frac{x+imx}{\sqrt{x^2+m^2x^2}} = \frac{1+im}{\sqrt{1+m^2}}$$

Therefore, the limit depends on m i.e. slope of the straight line. Thus, the limiting values is dependent on the path and the limit does not exists.

Example 4: Calculate the value $\lim\limits_{z\to\infty}\dfrac{iz^3+iz-1}{(2z+3i)(z-i)^2}$

Soln: $$\lim_{z\to\infty}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2} = \lim_{z\to\infty}\frac{z^3\left(i+\frac{i}{z^2}-\frac{1}{z^3}\right)}{z\left(2+\frac{3i}{z}\right)z^2\left(1-\frac{1}{z}\right)^2} = \frac{i}{2}$$

Differentiability of complex function:

$$f'(z) = \underset{\delta z\to 0}{Lt}\frac{[f(z+\delta z) - f(z)]}{\delta z}$$

The function will be differentiable if limit exists i.e the limiting value will be independent of path along with $\delta z \to 0$.

Example: $f(z) = (4x+y) + i(4y-x) \Rightarrow u = (4x+y)$ and $v = (4y-x)$

$\Rightarrow f(z+\delta z) = 4(x+\delta x) + (y+\delta y) + i\,[4(y+\delta y) - (x+\delta x)]$

$\Rightarrow f(z+\delta z) - f(z) = 4\delta x + \delta y + i(4\delta y - \delta x)$

$\Rightarrow \dfrac{f(z+\delta z)-f(z)}{\delta z} = \dfrac{4\delta x+\delta y+i(4\delta y-\delta x)}{\delta z} \Rightarrow \dfrac{\delta f}{\delta z} = \dfrac{4\delta x+\delta y-i\delta x+4i\delta y}{\delta x+i\delta y}$

Along real axis : $\delta x = \delta z$, $\delta y = 0$, $\Rightarrow \dfrac{\delta f}{\delta z} = 4 - i$

Along imaginary axis : $i\delta y = \delta z$, $\delta x = 0$, $\Rightarrow \dfrac{\delta f}{\delta z} = 4 - i$

Along a line : $y = x$, $\delta y = \delta x$, $\delta z = (1+i)\,\delta x$, $\Rightarrow \dfrac{\delta f}{\delta z} = \dfrac{5\delta x+3i\delta x}{(1+i)\delta x} = \dfrac{5+3i}{1+i} = 4 - i$

Therefore, $f(z) = (4x+y) + i(4y-x)$ will be differentiable.

5.3 Complex Analytic Function

A function $f(z)$ is said to be analytic at a point $z = z_0$ if it is single valued and has the derivative at every point in some neighhourhood of z_0. The function $f(z)$ is said to be analytic in a domain D if it is single valued and is differentiable at every point of domain D.

Cauchy Reamann Equations:

For a function $f(z) = u + iv$ to be analytic at all points in some region R,

Necessary conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient Condition: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y.

Derivative of $f(z)$**:** $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Example 5: Check whether $f(z) = \sin z$ is analytic or not.

Soln: $f(z) = \sin z = \sin(x+iy) = \sin x.\cos(iy) + \cos x.\sin(iy) = \sin x.\cosh y + i\cos x.\sinh y$

Therefore, u = sinx.coshy and v = cosx.sinhy

$\frac{\partial u}{\partial x} = \cos x.\cosh y; \frac{\partial u}{\partial y} = \sin x.\sinh y; \frac{\partial v}{\partial x} = -\sin x.\sinh y; \frac{\partial v}{\partial y} = \cos x.\cosh y$

So, C-R equation is satisfied, given f(z) is analytic.

Example 6: If the real part of a complex analytic function is $u(x,y) = x + \frac{1}{2}(x^2 - y^2)$, find the corresponding imaginary part.

Soln: $\frac{\partial u}{\partial x} = x + 1 = \frac{\partial v}{\partial y} \qquad \Rightarrow v(x,y) = y + xy + f(x)$

$\frac{\partial u}{\partial y} = -y = -\frac{\partial v}{\partial x} \qquad \Rightarrow v(x,y) = xy + g(y)$

Therefore, the imaginary part will be $v(x,y) = y + xy + C \quad (C = \text{numerical constant})$

Example 7: If $f(x,y) = (1+x+y)(1+x-y) + a(x^2 - y^2) - 1 + 2iy(1-x-ax)$ is a complex analytic function then find the value of a.

Soln: $u(x,y) = (1+x+y)(1+x-y) + a(x^2-y^2) - 1 \qquad \Rightarrow \frac{\partial u}{\partial x} = 2x + 2 + 2ax$

$v(x,y) = 2y(1-x-ax) \qquad \Rightarrow \frac{\partial v}{\partial y} = 2(1-x-ax)$

According to Cauchy Reamann equation, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 4x = -4ax \Rightarrow a = -1$

Example 8: The harmonic conjugate function of $u(x,y) = 2x(1-y)$ corresponding to a complex analytic function $\omega = u(x,y) + iv(x,y)$ is given $v(x,y) = \alpha x^2 + \beta y + \gamma y^2$ (Taking the integration constant to be zero). Which of the following statement is true ?

(a) $\alpha - \gamma = \beta$ (b) $\alpha + \gamma + \beta = 0$ (c) $\alpha + \gamma = \beta$ (d) $\alpha\gamma\beta = 1$

Soln: $u(x,y) = 2x(1-y)$

$\Rightarrow \frac{\partial u}{\partial x} = 2(1-y) = \frac{\partial v}{\partial y} \Rightarrow v = 2y - y^2 + f_1(x)$

$\Rightarrow \frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x} \Rightarrow v = x^2 + f_2(y)$

Therefore, the imaginary part of the complex function $v = x^2 - y^2 + 2y$

Comparing with the question, $\alpha = 1, \beta = 2, \gamma = -1 \quad \Rightarrow \alpha - \gamma = \beta$

Correct answer is (a)

Cauchy Reamann equations in Polar co-ordiantes:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Derivative of $f(z)$**:** $f'(z) = (\cos\theta - i\sin\theta)\frac{\partial f}{\partial r} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial f}{\partial \theta}$

Harmonic Function

Any function which satisfies the Laplace's equation, is known as harmonic function. If u + iv is an analytic function, then u, v are conjugate harmonic functions i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Example 9: Find the values of m, n such that $f(x,y) = x^2 + mxy + ny^2$ is harmonic in nature.

Soln: Since, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{dy^2} = 0 \Rightarrow 2n + 2 = 0 \Rightarrow n = -1$; '$m$' can take any value.

Method for finding conjugate Function:

Case 1: $f(z) = u + iv$, and u is known.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \Rightarrow v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

Case 2: $f(z) = u + iv$, and v is known

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \Rightarrow u = \int \frac{\partial v}{\partial y}dx - \int \frac{\partial v}{\partial x}dy$$

Example 10: Find the imaginary part of the complex analytic function whose real part is $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Soln: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \frac{\partial v}{\partial y} \Rightarrow v = 3x^2y - y^3 + 6xy + f_1(x)$

$\frac{\partial u}{\partial y} = -6xy - 6y = -\frac{\partial v}{\partial x} \Rightarrow v = 3x^2y + 6xy + f_2(y)$

$v(x,y) = 3x^2y - y^3 + 6xy + C$

Milne-Thomson Method : (To find Analytic function if either 'u' or 'v' is given)

Case 1: When 'u' is given,

(1) Find $\frac{\partial u}{\partial x} = \phi_1(x, y)$ and $\frac{\partial u}{\partial y} = \phi_2(x, y)$

(2) Replace x by z and y by 0 in $\phi_1(x, y)$ and $\phi_2(x, y)$ to get $\phi_1(z, 0)$ and $\phi_2(z, 0)$.

(3) Find $f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\}dz + c$

Case 2: When 'v' is given,

(1) Find $\frac{\partial v}{\partial x} = \psi_2(x, y)$ and $\frac{\partial v}{\partial y} = \psi_1(x, y)$

(2) Replace x by z and y by 0 in $\psi_1(x, y)$ and $\psi_2(x, y)$ to get $\psi_1(z,0)$ and $\psi_1(z,0)$.

(3) Find f(z) = $\int \{\psi_1(z,0) + i\psi_2(z,0)\}dz + c$

Example 11: Find the analytical function whose imaginary part is $v(x, y) = e^x(x\cos y - y\sin y)$

Soln: $\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \Psi_2(x, y)$ $\quad \Rightarrow \frac{\partial v}{\partial y} = -e^x x\sin y - e^x(\sin y + y\cos y) = \Psi_1(x, y)$

$\Psi_1(z,0) = 0$ and $\Psi_1(z,0) = e^z z + e^z$ $\quad \Rightarrow f(z) = \int 0 + i\left[e^z z + e^z\right]dz = ize^z + C$

Example 12: If the real part of a complex analytic function f(z) is given as, $u(x, y) = e^{-2xy}\sin(x^2 - y^2)$, then f(z) can be written as

(a) $ie^{iz^2} + C$ (b) $-ie^{iz^2} + C$ (c) $-ie^{-iz^2} + C$ (d) $ie^{-iz^2} + C$

Soln: $u(x, y) = e^{-2xy}\sin(x^2 - y^2)$

$$\Rightarrow \frac{\partial u}{\partial x} = e^{-2xy}(-2y)\sin(x^2 - y^2) + e^{-2xy}\cos(x^2 - y^2)2x = \phi_1(x, y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^{-2xy}(-2x)\sin(x^2 - y^2) + e^{-2xy}\cos(x^2 - y^2)(-2y) = \phi_2(x, y)$$

$$\therefore \quad \phi_1(z,0) = \cos z^2.2z\,,\ \phi_2(z,0) = \sin z^2(-2z)$$

$$\therefore \quad f(z) = \int\left(\cos z^2.2z - i\sin z^2.(-2z)\right)dz + c = 2\int\left(\cos z^2 + i\sin z^2\right).zdz + c$$

$$= 2\int e^{iz^2}.zdz + c = -i\,e^{iz^2} + c$$

Correct option is (b)

5.4 Power Series Expansion of Complex Function

Every analytic function which is analytic at $z = z_0$ can be expanded into power series about $z = z_0$.

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots\ldots$$

where, z_0 is the centre of power series.

For every power series there are three possibilities regarding the region of convergence of power series.

(i) The series converges within a disc

Example : $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots\ldots.$

Here $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{z^{n+1}}{z^n}\right| = |z|$; so for $|z| < 1$ the series will converge i.e. it will converge within the circle centered at the origin and of radius 1 unit.

(ii) The power series converges in the whole complex plane.

Example : $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots.$

Here, $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{z^{n+1}}{(n+1)!}\frac{n!}{z^n}\right| = \frac{|z|}{n+1} \to 0$ as $n \to \infty$ for all values of z i.e. the series will converge in the entire complex plane.

(iii) The power series converges for a particular values of z.

Example: $\sum_{n=0}^{\infty} n!z^n = 1 + z + 2z^2 + 6z^3 + \ldots\ldots\ldots.$

Here, $\left|\frac{t_{n+1}}{t_n}\right| = \left|\frac{(n+1)!z^{n+1}}{n!\,z^n}\right| = (n+1)|z| \to \infty$ as $n \to \infty$ for all values of z except $z = 0$ i.e. the series will converge only for $z = 0$.

Radius of convergence of power series:

Consider a circle centered at $z = z_0$ and radius r i.e. $|z - z_0| = R$, such that the power series is convergent for all points of the region $|z - z_0| < R$ (i.e. within the circle) and divergent for all points of the region $|z - z_0| > R$ (outside the circle). Therefore, R is said to be the radius of convergence of power series and defined as

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Example 14: Calculate the radius of convergence and region of convergence for the following power series

$$\sum_{n=0}^{\infty}\frac{(2n+3)}{(2n+5)(n+5)}z^n$$

Soln: $$R=\lim_{n\to\infty}\left|\frac{2n+3}{(2n+5)(n+5)}\frac{(2(n+1)+5)(n+1+5)}{2(n+1)+3}\right|=\lim_{n\to\infty}\left|\frac{\left(2+\frac{3}{n}\right)\left(2+\frac{7}{n}\right)\left(1+\frac{6}{n}\right)}{\left(2+\frac{5}{n}\right)\left(1+\frac{5}{n}\right)\left(2+\frac{5}{n}\right)}\right|=1$$

Region of convergence $|z|<1$

This series is convergent within a circle of radius 1 and centre (0, 0).

Example 15: Prove that the series

$$1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2+.....$$

has unit radius of convergence.

Soln: Neglecting the first term,

$$a_n=\frac{a(a+1)..(a+n-1)b(b+1)...(b+n-1)}{1.2...n.c(c+1)....(c+n-1)}$$

$$a_{n+1}=\frac{a(a+1)...(a+n-1)(a+n)(b+1)....(b+n-1)(b+n)}{1.2...n(n+1)c(c+1)....(c+n-1)(c+n)}$$

Dividing, $$\frac{a_{n+1}}{a_n}=\frac{(n+a)(n+b)}{(n+1)(c+n)}=\frac{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{(1+0)(1+0)}{(1+0)(1+0)}=1 \text{ or } R=1$$

Example 16: Find the radius of convergence of the series

$$\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+......$$

Soln: The coefficient of z^n of the given power series is given by

$$a_n=\frac{1.3.5...(2n-1)}{2.5.8...(3n-1)}$$

$$a_{n+1}=\frac{1.3.5...(2n-1)(2n+1)}{25.8...(3n-1)(3n+2)}$$

So, $$\frac{a_{n+1}}{a_n}=\frac{2n+1}{3n+2}=\frac{2}{3}.\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}$$

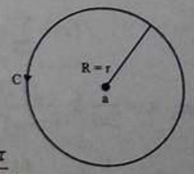
Therefore, $$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{2}{3}.\frac{(1+0)}{(1+0)}=\frac{2}{3}\Rightarrow R=\frac{3}{2}$$

Taylor Series Expansion

If a function f(z) is analytic at all points inside and on a circle C, having center at $z=a$ and radius r, then at each point z inside C, the function f(z) can be expanded as

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+....+\frac{f^n(a)}{n!}(z-a)^n+....$$

$$\Rightarrow f(z)=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n$$



Example 17: Expand the function $f(x)=\sin x$ about the point $x=\frac{\pi}{6}$.

Soln: $$f(x)=f\left(\frac{\pi}{6}\right)+\left(x-\frac{\pi}{6}\right)f'\left(\frac{\pi}{6}\right)+\frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2f''\left(\frac{\pi}{6}\right)+......$$

$$=\sin\frac{\pi}{6}+\left(x-\frac{\pi}{6}\right)\cos\frac{\pi}{6}+\frac{\left(x-\frac{\pi}{6}\right)^2}{2!}\left(-\sin\frac{\pi}{6}\right)+........$$

$$=\frac{1}{2}+\left(x-\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)^2\frac{1}{2}-\frac{1}{6}\left(x-\frac{\pi}{6}\right)^3\frac{\sqrt{3}}{2}+......$$

$$f(x)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{1}{4\sqrt{3}}\left(x-\frac{\pi}{6}\right)^3+....$$

Example 18: Expand the following function $f(z)=\frac{1}{(z-1)(z-4)}$ about z = 2.

Soln: $$\frac{1}{(z-1)(z-4)}=\frac{1}{3}\frac{1}{z-4}-\frac{1}{3}\frac{1}{z-1}=\frac{1}{3}\frac{1}{(z-2)-2}-\frac{1}{3}\frac{1}{(z-2)+1}$$

$$=\frac{1}{3(-2)}\frac{1}{\left[1-\frac{z-2}{2}\right]}-\frac{1}{3}\frac{1}{[1+(z-2)]}=-\frac{1}{6}\left[1-\frac{z-2}{2}\right]^{-1}-\frac{1}{3}[1+(z-2)]^{-1}$$

$$=-\frac{1}{6}\left[1+\frac{z-2}{2}+\frac{(z-2)^2}{4}+\frac{(z-2)^3}{8}\ldots\ldots\right]-\frac{1}{3}\left[1-(z-2)+(z-2)^2-(z-2)^3+\ldots\ldots\right]$$

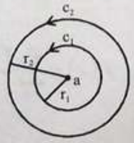
$$=-\frac{1}{2}+\frac{1}{4}(z-2)-\frac{3}{8}(z-2)^2+\ldots\ldots\ldots$$

Laurent Series expansion:

Laurent's Theorem:

If $f(z)$ is analytic inside and on the boundary of the ring-shaped region R bounded by two concentric circles C_1 and C_2 with centre at $z=a$ and respective radii r_1 and r_2 $(r_2>r_1)$, then for all z in the region R the Laurent series expansion of $f(z)$ about $z=a$ will be

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$



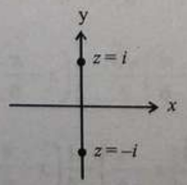
where, $a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$

$$b_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{(z-a)^{-(n+1)}}dz$$

Note: (i) The Laurrent series converges in the region R.

(ii) The point at which Laurrent series expansion has to be calculated, will be the centre of concentric circles.

Example 19: $f(z)=\frac{1}{1+z^2}$ about $z=i$



Soln: $f(z)=\frac{1}{1+z^2}=\frac{1}{(z+i)(z-i)}=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$

$$=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{(z-i)+2i}\right]$$

$$=\frac{1}{2i}(z-i)^{-1}-\frac{1}{2i}\left[\frac{1}{2i\left[1+\frac{(z-i)}{2i}\right]}\right]=\frac{1}{2i}(z-i)^{-1}-\frac{1}{4i^2}\left[1-\frac{(z-i)}{2i}+\frac{(z-i)^2}{4i^2}\ldots\ldots\right]$$

$$=\underbrace{\frac{1}{2i}(z-i)^{-1}}_{\text{Converges for }|z-i|\geq 0}+\underbrace{\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^n}}_{\text{Converges for }|z-i|<2}$$

So, the function will be convergent for $0\leq|z-i|<2$

Example 20: Exapand the function $f(z)=\frac{1}{(z-1)(z-2)}$ in the annulus $z=1$ and $z=2$.

Soln: $f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$ when $1<|z|<2$, then $\frac{1}{|z|}<1$ & $\frac{|z|}{2}<1$

$$f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right]-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\ldots\right]=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Example 21: Find the Laurent series of the function $f(z)=\frac{1}{z^2(1-z)}$ about z=0

Soln: $f(z)=\frac{1}{z^2(1-z)}=\frac{1}{z^2}(1-z)^{-1}=\frac{1}{z^2}(1+z+z^2+\ldots\ldots)=\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\ldots\ldots\ldots$

Example 22: Expand $\frac{1}{z^2-3z+2}$ for (i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

Soln: $\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}$

(i) $0<|z|<1$: $f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)^{-1}}-\frac{1}{-1(1-z)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1-z)^{-1}=\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{\infty}z^n$

(ii) $1<|z|<2$: then $\frac{1}{|z|}<1$ and $\frac{|z|}{2}<1$

$$\therefore\quad f(z)=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\ldots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\ldots\right)$$

(iii) $|z|>2$: then $\frac{2}{|z|}<1$ and $\frac{1}{|z|}<1$

$$\therefore\quad f(z)=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\ldots\ldots\ldots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\ldots\ldots\right)$$

5.5 Singularity of Complex Function

Singular points of a function:

The points at which the function ceases to be analytic, are said to be the singular points of the function.

Example: $f(z)=\frac{1}{(z-2)}$ has a singularity at z = 2.

The singularity will be of the following types:

Isolated singularity:

A point $z=z_0$ is said to be isolated singularity of *f* (*z*) if

(a) *f* (*z*) is not analytic at $z=z_0$

(b) *f* (*z*) is analytic in the neighbourhood of $z=z_0$ i.e. there exists a neighbourhood of $z=z_0$, containing no other singularity.

Example:

(i) Function $f(z)=\frac{1}{z}$ is analytic everywhere except at z = 0, therefore z = 0 is an isolated singularity.

(ii) The function $f(z)=\frac{z+2}{(z-1)(z-2)(z-3)}$ has three isolated singularities at $z = 1$, 2 and 3.

Non-isolated singularity:

A point $z=z_0$ is said to be non-isolated singularity of *f* (*z*) if

(a) *f* (*z*) is not analytic at $z=z_0$

(b) there exists a neighbourhood of $z=z_0$, containing other singularities of *f* (*z*).

Example: $f(z)=\frac{1}{\left[\sin\frac{\pi}{z}\right]}$

Condition of singularity: $\sin\frac{\pi}{z}=0\Rightarrow\frac{\pi}{z}=n\pi\Rightarrow z=\frac{1}{n}\,(n=0,1,2,.....\infty)$

The point $z=0$ corresponding to $n\to\infty$, will be surrounded by infinite many singular points of *f* (*z*). Thus, $z=0$ is an non-isolated singularity of *f* (*z*).

Types of isolated singularity

If *f* (*z*) is an isolated singular point at z = a, then we can expand *f* (*z*) about z = a into Laurent series as:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}}_{\text{principal part of the expansion}}=\left[a_0+a_1(z-a)+a_2(z-a)^2+.....\right]+\left[\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+.....\right]$$

Therefore, three types of singulatiry are as follows:

(1) Removable singularity:

If the principal part of the Laurrent series expansion of *f* (*z*) about z = a contains no term i.e. $b_n=0$ for all 'n', then *f* (*z*) has a removable singularity at z = a.

In this case, Laurent series expansion is $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$

Example: $f(z)=\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-.........\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-.....$

Since, there is no negative term in the laurent series expansion of *f* (*z*) about z = 0, hence z = 0 is a removable singularity of *f* (*z*).

(2) Non-essential singularity or Pole:

If the principal part of the Laurrent series expansion of *f* (*z*) about z = a contains finite number of terms, say *m*, i.e. $b_n=0$ for all $n>m$, then *f* (*z*) has a non-essential singularity or a pole of order *m* at z = a . A pole of order one is also known as simple pole.
Thus if z = a is pole of order m of function *f* (*z*), then *f* (*z*) will have the Laurent series expansion of the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{m}b_n(z-z_0)^{-n}$$

Example: $f(z)=\frac{z}{(z-1)(z+2)^2}$ has a simple pole at z = 1 and a pole of order 2 at z = -2.

(3) Essential singularity:

If the principal part of the Laurent series expansion of *f* (*z*) about z = a, contains infinite number of terms i.e. $b_n\neq 0$ for infinitely many values of n, then *f* (*z*) has an essential singularity at z = a.

Example: $f(z)=e^{1/z}=1+\frac{1}{z}+\frac{1}{z^2\,2!}+......$

Therefore, *f* (*z*) has an essential singularity at z = 0.

Example 23: Examine the nature of singularity of the functions: (a) $\sin\left(\frac{1}{1-z}\right)$, (b) $(z-3)\sin\left(\frac{1}{z+2}\right)$.

Soln: (a) $\sin\left(\frac{1}{1-z}\right)=\frac{1}{1-z}-\frac{1}{(1-z)^3.3!}+\frac{1}{(1-z)^5.5!}-...........$

so, z = 1 is an isolated essential singular point.

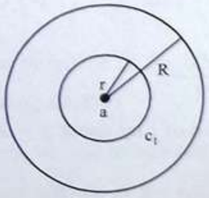
(b) $(z-3)\sin\left(\frac{1}{z+2}\right)=(z-3)\left[\frac{1}{z+2}-\frac{1}{(z+2)^3.3!}+\frac{1}{(z+2)^5.5!}-........\right]$

so, z = -2 is an isolated essential singular point.

5.6 Residue of a Complex Function

Residue at a pole:

Let, z = a be a pole of order 'm' of f (z) and C_1 is a circle of radius 'r' with center at z = a which does not contain singularities except z = a, then f(z) is analytic within the annular region r < | z – a | < R can be expanded into Laurent series within the annulur region as:



$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$$

Co-efficient b_1 is known as residue of f (z) at z = a i.e. Res. $f(z=a)=b_1=\frac{1}{2\pi i}\oint_{C_1}f(z)dz$

Methods of finding residues:

CASE 1: Residue at simple pole:

(a) Method 1: Res. $f(z=a)=\underset{z\to a}{\text{Lt}}\,(z-a)f(z)$

(b) Method 2: If $f(z)=\frac{\phi(z)}{\psi(z)}$ where $\psi(a)=0$ but $\phi(a)\neq 0$, then Res. $f(z=a)=\frac{\phi(a)}{\psi'(a)}$

CASE 2: Residue at a pole of order 'n':

(a) Method 1: Res. $f(z=a)=\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$

(b) Method 2: First put z + a = t and expand it into series, then Res. f (z = a) = co-efficient of 1/t.

CASE 3: Residue at z = ∞:

$$\text{Res.}f(z=\infty)=\underset{z\to\infty}{Lt}\left[-zf(z)\right]$$

Example 24: Find the singular points of the following function and the corresponding residues:

(a) $f(z)=\frac{1-2z}{z(z-1)(z-2)}$ (b) $f(z)=\frac{z^2}{z^2+a^2}$ (c) $f(z)=z^2e^{1/z}$

Soln: (a) $f(z)=\frac{1-2z}{z(z-1)(z-2)}\Rightarrow$ Poles : $z=0, z=1, z=2$

Res. $f(z=0)=\underset{z\to 0}{Lt}(z-0)f(z)=\underset{z\to 0}{Lt}\frac{1-2z}{(z-1)(z-2)}=\frac{1}{2}$

Res. $f(z=1)=\underset{z\to 1}{Lt}(z-1)f(z)=\underset{z\to 1}{Lt}\frac{1-2z}{z(z-2)}=1$

Res. $f(z=2)=\underset{z\to 2}{Lt}(z-2)f(z)=\underset{z\to 2}{Lt}\frac{1-2z}{z(z-1)}=-\frac{3}{2}$

(b) $f(z)=\frac{z^2}{z^2+a^2}\Rightarrow$ Poles : $z=ia, z=-ia$

Res. $f(z=ia)=\left(\frac{z^2}{2z}\right)_{z=ia}=\frac{1}{2}ia$; Res. $f(z=-ia)=\left(\frac{z^2}{2z}\right)_{z=-ia}=-\frac{1}{2}ia$

(c) $f(z)=z^2e^{1/z}=z^2\left[1+\frac{1}{z}+\frac{1}{z^2.2!}+\frac{1}{z^3.3!}+\dots\dots\right]\Rightarrow$ Poles : $z=0$

Res. $f(z=0)=$ Coefficient of $\frac{1}{z}=\frac{1}{3!}=\frac{1}{6}$

Cauchy's Residue Theorem:

If f (z) in single-valued and analytic in a simple closed curve 'C, except at a finite number of singular points within C, then

$$\oint_C f(z)dz=2\pi i\left(\text{sum of the residues at poles within 'C'}\right)$$

Example 25: Evaluate the integral: $\oint_C\frac{4-3z}{z(z-1)(z-3)}dz$ where $|z|=\frac{3}{2}$

Soln: $f(z)=\frac{4-3z}{z(z-1)(z-3)}\Rightarrow$ Poles : $z=0, z=1, z=3$

But, the given contour is circle centered at the origin and radius 3/2 units.

Therefore, only z = 0 and z = 1 within the contour.

$I=2\pi i\left[\text{Res.}f(z=0)+\text{Res.}f(z=1)\right]=2\pi i\left[\frac{4}{3}-\frac{1}{2}\right]=\frac{5\pi i}{3}$

Example 26: Evaluate the integral: $\oint_C\frac{e^{2z}+z^2}{(z-1)^5}dz$ where $|z|=2$

Soln: $f(z)=\frac{e^{2z}+z^2}{(z-1)^5}\Rightarrow$ Poles : $z=1$ (order 5)

$I=2\pi i\times\text{Res.}f(z=1)=2\pi i\times\frac{1}{4!}\frac{d^4}{dz^4}\left[e^{2z}+z^2\right]_{z=1}=2\pi i\times\frac{2e^2}{3}=\frac{4\pi ie^2}{3}$

Definite integrals of trigonometric functions of $\cos\theta$ and $\sin\theta$:

Method:

(i) Consider the contour to be a circle centered at the origin and having radius one unit i.e. $|z|=1$

(ii) Assume, $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$

(iii) Therefore, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$.

Replacements regarding $\cos\theta$ and $\sin\theta$ is to be done only in the denominator of the given integral.

(iv) The limit will be changed from $0 \to 2\pi$ to $\oint\limits_C$

(v) Find the singular points and find residues at only those singular points which lies inside the unit circle.

(vi) Finally use the Cauchy residue theorem.

Example 27: Evaluate the integral: $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}$; $a>b>0$

Soln: $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi}\frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)} = \int_0^{2\pi}\frac{2dz}{i(bz^2+2az+b)}$

The singular points are at $\Rightarrow$ $z=\alpha=\frac{-a+\sqrt{a^2-b^2}}{b}$ & $z=\beta=\frac{-a-\sqrt{a^2-b^2}}{b}$

The singular point $z=\beta$ will lie outside the unit circle as a>b>0 while the singular point $z=\alpha$ will lie inside the unit circle which is a simple pole.

$$\text{Res.} f(z=\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{2}{ib}\frac{(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{2}{ib(\alpha-\beta)} = \frac{2}{ib}\times\frac{b}{2\sqrt{a^2-b^2}} = \frac{1}{i\sqrt{a^2-b^2}}$$

Therefore, by cauchy Residue theorem.

$$I = 2\pi i\times\text{Residue} = 2\pi i\times\frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 28: Evaluate the integral $\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta}$

Soln: $\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta} = \oint\limits_C\frac{1}{5+4.\frac{1}{2}\left(z+\frac{1}{2}\right)}.\frac{dz}{iz} = \frac{1}{i}\oint\limits_C\frac{dz}{(5z+2z^2+2)} = \oint\limits_C\frac{dz}{(2z+1)(z+2)}$

$f(z)=\frac{1}{(2z+1)(z+2)}$ $\Rightarrow$ poles are at $z=-\frac{1}{2},-2$ (Only $z=-½$ is within the circle)

$$\text{Res. } f(z=-½) = \left(z+\frac{1}{2}\right).\frac{1}{(2z+1)(z+2)}\bigg|_{z=-\frac{1}{2}} = \frac{1}{2(z+2)}\bigg|_{z=-\frac{1}{2}} = \frac{1}{3}$$

Hence, the integral $=\frac{2\pi i}{3}$

Example 29: Evaluate the integral $\int_0^{2\pi}\frac{d\theta}{1-2m\cos\theta+m^2},(m^2<1)$

Soln: $\int_0^{2\pi}\frac{d\theta}{1-2m\cos\theta+m^2},(m^2<1) = \oint\limits_C\frac{dz/iz}{1-2m\times\frac{1}{2}\left[z+\frac{1}{z}\right]+m^2} = \oint\limits_C\frac{dz}{iz\left(z-m\left(\frac{z^2+1}{z}\right)+m^2z\right)}$

$$=\oint\limits_C\frac{dz}{i\left(z-mz^2-m+m^2z\right)} = \oint\limits_C\frac{1}{i}\frac{dz}{(z-m)-mz(z-m)} = \frac{1}{i}\oint\limits_C\frac{dz}{(z-m)(1-mz)}$$

$f(z)=\frac{1}{(z-m)(1-mz)}$ has poles of order one at $z=m,\frac{1}{m}$

Since, $m<1$ so $z=m$ will be within the circle $|z|=1$

So, $I=\frac{1}{i}\left[2\pi i\times res.f(z=m)\right] = \frac{1}{i}\left[2\pi i\times\lim_{z\to m}(z-m)\frac{1}{(z-m)(1-mz)}\right]$

$$=\frac{1}{i}\times2\pi i\times\frac{1}{1-m^2} = \frac{2\pi}{1-m^2}$$

Example 30: Evaluate the integral $\int_0^{2\pi}e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta$

Soln: $\int_0^{2\pi}e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta$ = Real part of $\int_0^{2\pi}e^{\cos\theta}e^{i(\sin\theta-n\theta)}d\theta$

= Real part of $\int_0^{2\pi}e^{(\cos\theta+i\sin\theta)}e^{-in\theta}d\theta$ = Real part of $\oint\limits_C e^z.z^{-n}\frac{dz}{iz}$

= Real part of $\frac{1}{i}\oint\limits_C\frac{e^z}{z^{n+1}}dz$

$f(z)=\frac{e^z}{z^{n+1}}$ $\Rightarrow$ poles are at $z=0$ of order $n \Rightarrow$ Res. $f(z=0)=\frac{1}{n!}.\frac{d^n}{dz^n}\left(z^{n+1}.\frac{e^z}{z^{n+1}}\right)\bigg|_{z=0} = \frac{1}{n!}$

Hence the given integral = real part of $\left(\frac{1}{i}.2\pi i.\frac{1}{n!}\right)=\frac{2\pi}{n!}$

Evaluation of improper integrals between the limit $-\infty$ to $+\infty$:

Theorem 1:

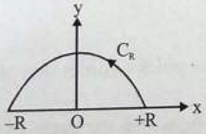
If AB is an arc $(\alpha \le \theta \le \beta)$ of a circle $|z| = R$ and $\lim\limits_{z\to\infty} z\, f(z) = k(\text{constant})$. Then,

$$\lim_{R\to\infty} \int_{AB} f(z)\,dz = i(\beta-\alpha)k$$

Theorem 2:

If f(z) is a function of complex variable z which satisfies the following conditions:
(i) f(z) is analytic in the upper half plane except at a finite number of poles.
(ii) f(z) has no poles on the real axis.
(iii) $z\,f(z) \to 0$ uniformly as $|z| \to \infty$

So that $\lim\limits_{R\to\infty} \int_{C_R} f(z)\,dz \to 0$

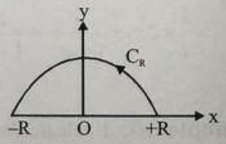


Theorem 3:

Jordan's Lemma: If f(z) is a function of complex variable z which satisfies the following conditions.
(i) f(z) is analytic in the upper half plane except at a finite number of poles.
(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \le \arg z \le \pi$, then

$$\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)\,dz \to 0$$

where m is a +ve number and C_R is the semicircle of radius R.



Theorem 4:

If AB is an arc $(\theta_1 \le \theta \le \theta_2)$ of a circle $|z-a| = r$ and $\underset{z\to a}{Lt}(z-a)f(z) = k$, then

$$\lim_{r\to 0} \int_{AB} f(z)\,dz = i(\theta_2-\theta_1)k$$

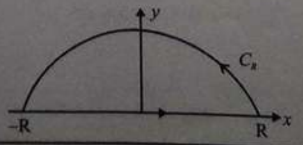
Short Method

Case 1:

If $f(x)$ contains only polynomial terms, then we will take $f(z)$ same as given as $f(x)$. Then find the singular points of $f(z)$ and check which points lie in the upper half plane.

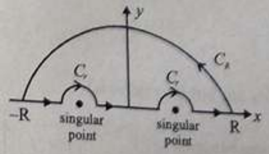
(A) If the singular points does not lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i[\Sigma\text{ Res. at all poles within C}] - \left[\lim_{z\to\infty} z\, f(z)\right] i\pi$$



(B) If the singular point lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \pi i[\Sigma\text{ Res. at all poles within C}] - \left[\lim_{z\to\infty} z\, f(z)\right] i\pi$$



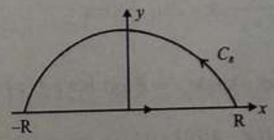
Case 2:

If $f(x)$ contains cosines and sine functions along with polynomial functions then $f(x)$ can be treated as a real or imaginary part of $f(z)$. Then find the singular points of $f(z)$ and check which points lie in the upper half plane.

(A) If the singular points does not lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i[\Sigma\text{Res. at all poles within C}] \text{ if as } z\to\infty,\ \frac{f(z)}{e^{imz}} \to 0 \text{ and}$$

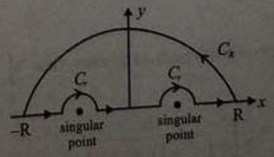
$$\int_{-\infty}^{\infty} f(x)\,dx = \text{Real part or imaginary part of} \int_{-\infty}^{\infty} f(z)\,dz$$



(B) If the singular points lie on the real axis, then

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi i[\Sigma\text{ Res. at all poles within C}] \text{ if as } z\to\infty,\ \frac{f(z)}{e^{imz}} \to 0 \text{ and}$$

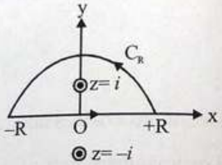
$$\int_{-\infty}^{\infty} f(x)\,dx = \text{Real part or imaginary part of} \int_{-\infty}^{\infty} f(z)\,dz$$



Example 31: Calculate the following integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Soln: $f(x) = \frac{1}{1+x^2}$

$f(z) = \frac{1}{1+z^2}$ has poles at $z = \pm i$ (Poles are not on real axis)



In this case we will take a contour 'C' consisting of

(i) A semicircle $C_R : |z| = R$ in the upper half of complex plane.

(ii) Real axis from –R to +R

$\oint f(z)dz = 2\pi i$ [sum of residues at the poles]

$$\Rightarrow \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i(\text{residue at } z = +i) \qquad \text{... (i)}$$

Residue of $f(z)$ at $z = +i = \lim_{z \to +i}(z-i)\frac{1}{1+z^2} = \frac{1}{2i}$

Taking the limit $R \to \infty$ in the equation (i), we get

$$\Rightarrow \lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^2}dz + \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = 2\pi i\left(\frac{1}{2i}\right) = \pi \qquad \text{... (ii)}$$

C_R is an arc $0 \le \theta \le \pi$ of a circle $|z| = R$ and $\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty}\frac{z}{1+z^2} = \lim_{z\to\infty}\frac{1/z}{1/z^2+1} = 0$

Thus, $\lim_{R\to\infty}\int_{C_R} f(z)dz = i(\pi - 0)(0) = 0$

Therefore, from equation (ii) $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \pi$

Short Method:

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{\infty}\frac{dz}{1+z^2} = 2\pi i[\text{Res. } f(z=i)] - \lim_{z\to\infty}[zf(z)]i\pi = 2\pi i\left[\frac{1}{2i}\right] = \pi$$

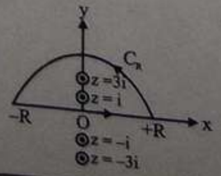
Example 32: Apply calculus of residues to show that $\int_0^\infty \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{24}$

Soln: $I = \int_0^\infty \frac{dx}{(x^2+1)(x^2+9)} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+9)}$

Let $f(z) = \frac{1}{(z^2+1)(z^2+9)}$

The singular points of f(z) are

$(z^2+1)(z^2+9) = 0 \quad \Rightarrow \quad z = i, -i, 3i, -3i$



The only poles lying within the contour (upper half plane) are at z = +i and +3i

$$\int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i(\text{Res. at } (z=i) + \text{Res. at } (z=3i)) - \lim_{z\to\infty}[zf(z)]i\pi$$

Here, $\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{z}{(z^2+1)(z^2+9)} = \lim_{z\to 0}\frac{z}{z^4\left(1+\frac{1}{z^2}\right)\left(1+\frac{9}{z^2}\right)} = 0$

Therefore, $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+9)} = 2\pi i\left[\frac{1}{16i} - \frac{1}{48i}\right] = \frac{\pi}{12}$

$$\Rightarrow \int_0^\infty \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{24}$$

Example 33: Prove $\int_{-\infty}^{\infty}\frac{\cos x dx}{(x^2+a^2)} = \frac{\pi}{a}e^{-a} \quad ; a > 0$

Soln: Let $f(z) = \frac{e^{iz}}{(z^2+a^2)}$; For singuarities, $z^2 + a^2 = 0 \Rightarrow \qquad z = \pm ia$.

The only pole which lies in the upper half of the circle is at z = ia.

Therefore, Residue at z = ia= $\lim_{z\to ia}\frac{(z-ia)e^{iz}}{(z-ia)(z+ai)} = \frac{e^{-a}}{2ia}$

Now by cauchy residue theorem.

$$\int_{C_R} f(z)dz + \int_{-R}^{+R} f(x)dx = 2\pi i(\text{sum of the residues at the poles}) \qquad \text{... (1)}$$

Now in the limit of $R \to \infty$, equation (2) becomes

$$\underset{R\to\infty}{Lim}\int_{C_R} f(z)dz + \int_{-\infty}^{+\infty} f(x)dx = 2\pi i(\text{sum of the residues at the poles}) \qquad \text{... (2)}$$

As $z \to \infty$, $\frac{1}{z^2+a^2} \to \infty$

Therefore, By Jordan's lemma $\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{(z^2+a^2)}dz \to 0$

Therefore, equation (2) gives

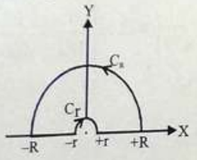
$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = 2\pi i \times \frac{e^{-a}}{2ia} \Rightarrow \int_{-\infty}^{\infty}\frac{e^{ix}dx}{x^2+a^2} = \frac{\pi}{a}e^{-a} \Rightarrow \int_{-\infty}^{\infty}\frac{(\cos x + i\sin x)}{(x^2+a^2)} = \frac{\pi}{a}e^{-a}$$

$$\Rightarrow \quad \int_0^\infty \frac{\cos x}{(x^2+a^2)}dx = \frac{\pi}{2a}e^{-a}$$

Evaluation of some improper integrals in which the pole lies on the real axis:

Example 34: Prove $\int_{-\infty}^{\infty}\frac{\sin mx}{x}dx=\frac{\pi}{2};\ m>0$

Soln: Let $f(z)=\frac{e^{imz}}{z}$. The singularity is at z=0 which is at real axis. No pole will be inside the semicircle. Now taking the integral counter clockwise and using the cauchy integral theorem, we get



According to Cauchy Residue theorem, we get

$$\int_{C_R}f(z)dz+\int_{-R}^{-r}f(x)dx+\int_{C_r}f(z)dz+\int_{r}^{R}f(x)dx=0 \text{ (Since there is no pole inside the contour)}$$

Taking the limit $R\to\infty$ and $r\to 0$, we get

$$\lim_{R\to\infty}\int_{C_R}f(z)dz+\int_{-\infty}^{0}f(x)dx+\lim_{r\to 0}\int_{C_r}f(z)dz+\int_{0}^{\infty}f(x)dx=0 \qquad \dots(1)$$

Here, C_r is arc of the circle $|z|=r$ between $(\pi\le\theta\le 0)$, then according to theorem (4)

Now, $\lim_{r\to 0}\int_{C_r}f(z)dz=i(0-\pi)\times\lim_{z\to 0}(z-0)f(z)=-i\pi$

As $z\to\infty, \frac{1}{z}\to 0$, therefore, $\lim_{R\to\infty}\int_{C_R}\frac{e^{imz}}{z}dz=0$ (By Jordan's lemma)

Equation (1) becomes, $0+\int_{-\infty}^{0}f(x)dx-\pi i+\int_{0}^{\infty}f(x)dx=0\Rightarrow\int_{-\infty}^{\infty}f(x)dx=\pi i\Rightarrow\int_{-\infty}^{\infty}\frac{e^{imx}}{x}dx=\pi i$

Equating real and imaginary parts, we have

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x}dx=0 \quad \& \quad \int_{-\infty}^{\infty}\frac{\sin mx}{x}dx=\pi \quad\Rightarrow 2\int_{0}^{\infty}\frac{\sin mx}{x}dx=\pi \quad\Rightarrow\int_{0}^{\infty}\frac{\sin mx}{x}dx=\pi/2$$

If $m=1 \quad\Rightarrow \quad \int_{0}^{\infty}\frac{\sin x}{x}dx=\pi/2=\int_{0}^{\infty}\frac{\sin x}{x}dx$

Example 35: If $a>0$, $m>0$, then

$$\int_{0}^{\infty}\frac{\cos mx dx}{\left(x^2+a^2\right)^2}=\frac{\pi}{4a^3}(1+ma)e^{-ma}$$

Soln: $f(z)=\frac{e^{imz}}{\left(a^2+z^2\right)^2}$ has poles of order 2 at $z=\pm ia$

Only $z=ia$ is within the contour.

As $z\to\infty$, $f(z)/e^{imz}=\frac{1}{\left(z^2+a^2\right)}\to 0$

$$\int_{-\infty}^{\infty}\frac{e^{imz}}{\left(z^2+a^2\right)^2}dx=2\pi i\left[\text{Res}(z=ia)\right]=2\pi ie^{-ma}\frac{(1+ma)}{4a^3i}=\frac{\pi}{2a^3}e^{-ma}(1+ma)$$

Equating real parts from both sides,

$$\int_{-\infty}^{\infty}\frac{\cos mx dx}{\left(x^2+a^2\right)^2}=\frac{\pi}{2a^3}(1+ma)e^{-ma}$$

But $\int_{-\infty}^{\infty}\frac{\cos mx dx}{\left(x^2+a^2\right)^2}dx=2\int_{0}^{\infty}\frac{\cos mx}{\left(x^2+a^2\right)^2}dx$

Hence, the last gives $\int_{0}^{\infty}\frac{\cos mx}{\left(x^2+a^2\right)^2}dx=\frac{\pi}{4a^3}(1+ma)e^{-ma}$

Deductions. (i) Putting m = 1, we get, $\int_{0}^{\infty}\frac{\cos x dx}{\left(x^2+a^2\right)^2}=\frac{\pi}{4a^3}(1+a)e^{-a}$

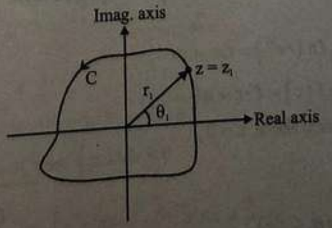
(ii) Taking m = a = 1, we obtain, $\int_{0}^{\infty}\frac{\cos x}{\left(x^2+1\right)^2}dx=\frac{\pi}{4}.2e^{-1}=\frac{\pi}{2e}$

5.7 Branch Point Singularity:

Suppose the following multi-valued function

$$f(z)=\sqrt[5]{z}$$

i.e. corresponding to a single-value of z, we will get five different values of $f(z)$



We know, $Z=re^{i\theta}\Rightarrow f(z)=\sqrt[5]{z}=r^{1/5}e^{i\theta/5}$

Suppose at $z=z_1=r_1e^{i\theta_1}$, $f(z)=r_1^{1/5}e^{i\theta_1/5}$

When we move around C by one complete cycles around z = 0, then θ increases from θ_1 to $\theta_1+2\pi$ and we are at same point $z=z_1$. But the values of $f(z)$ will be $r_1^{1/5}e^{i(\theta_1+2\pi)/5}$

After two complete cycles about z = 0, the point $z=z_1$ will be same but $f(z)=r_1^{1/5}e^{i(\theta+4\pi)/5}$

After 5 complete cycles about z = 0, the value of $f(z)$ will be repeated.

i.e. for a single value of z, there will be five different values of $f(z)$. So, $f(z)$ will be a five valued function of z.

$$f(z)=\sqrt[5]{z}=\left(re^{i(\theta+2k\pi)}\right)^{1/5}=r^{1/5}e^{i(\theta+2k\pi)/5} \qquad (k=0,1,2,3,4)$$

The five different values of $f(z)$ are said to be the branches of the multivalued function $f(z)$. The corresponding ranges of θ will be

$$0\le\theta<2\pi,\ 2\pi\le\theta<4\pi,\ 4\pi\le\theta<6\pi,\ 6\pi\le\theta<8\pi,\ 8\pi\le\theta<10\pi$$

The first interval, $0\le\theta<2\pi$ is also called principal branch of θ and corresponding value of $f(z)$ will be the principal branch of the function $f(z)$.

Since, different values of $f(z)$ can be found out by successively going around/encircling z = 0, then z = 0 is known as branch point of $f(z)$

Characteristics of branch points:

(i) At branch points, the multi-valued functions $f(z)$ is not differentiable i.e. $f(z)$ is not analytic at branch points.

(ii) At branch points, the values of branches of $f(z)$ are equal.

Example: $f(z)=\sqrt[2]{z^2+1}=\sqrt[2]{(z+i)(z-i)}$

$$\Rightarrow \quad f'(z)=\frac{1}{2\sqrt{z-i}}\sqrt{z+i}+\frac{1}{2\sqrt{z+i}}\sqrt{z-i}$$

The function is a double valued function of z.

The function is not differentiable at $z=\pm i$ i.e. not analytic.

Therefore, $z=\pm i$ are two branch points of $f(z)$

At $z=\pm i$, the value of two branches of $f(z)$ are equal to zero.

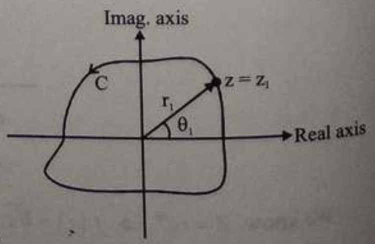
Example : $f(z)=\ell n\, z=\ell n\left(re^{i\theta}\right)=\ell nr+i\theta$

Suppose at $z=z_1=r_1e^{i\theta_1}$, $f(z)=\ell nr_1+i\theta_1$

After one complete revolution about $z=0$,
We are returning to

$$z=z_1=r_1e^{i(\theta_1+2\pi)}$$

Such that, $f(z)=\ell nr_1+i(\theta_1+2\pi)$



After two complete revolution about z = 0 at $z=z_1=r_1\,e^{i(\theta+4\pi)}$

$$f(z)=\ell nr_1+i(\theta_1+4\pi) \text{ and so on.}$$

Therefore, $f(z)=\ell nz$ has infinite branches and it is a infinite valued function of z.

Since, the function $f(z)=\ell nz$ is not differentiable or analytic at z = 0, therefore z = 0 is a branch point of $f(z)$.

CSIR-NET/JRF Previous Years Questions with Solutions

1. Which of the following is an analytic function of the complex variable z = x + iy in the domain |z| < 2? **[CSIR June 2011]**

(a) $(3+x-iy)^7$

(b) $(1+x+iy)^4(7-x-iy)^3$

(c) $(1-2x-iy)^4(3-x-iy)^3$

(d) $(x+iy-1)^{1/2}$

Soln. Any complex function which contains z will be analytic in nature whereas any complex function which contains $\bar{z}$, Re (z), Im(z) will not be analytic in nature.

$$(3+x-iy)^7=(3-\bar{z})^7$$

$$(1+x+iy)^4(7-x-iy)^3=(1+z)^4(7-z)^3$$

$$(1-2x-iy)^4(3-x-iy)^3=(1-\text{Re}(z)-z)^4(3-z)^3$$

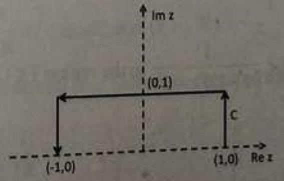
$$(x+iy-1)^{1/2}=(z-1)^{1/2}$$

The function in the fourth option is a multivalued complex function having a branch point at z = 1.

Therefore, only the complex function $(1+x+iy)^4(7-x-iy)^3$ is analytic in nature.

Correct answer is (b)

2. The value of the integral $\int_C dz\, z^2e^z$, where C is an open contour in the complex z-plane as shown in figure below: **[CSIR June 2011]**



(a) $\frac{5}{e}+e$

(b) $e-\frac{5}{e}$

(c) $\frac{5}{e}-e$

(d) $-\frac{5}{e}-e$

Soln. Since z^2e^z is an analytic function, then $\oint z^2e^z dz=0$.

$$\Rightarrow \int_C z^2e^z dz+\int_{-1}^{+1} z^2e^z dz=0 \quad \Rightarrow \int_C z^2e^z dz=-\int_{-1}^{+1} x^2e^x dx=\frac{5}{e}-e$$

Correct answer is (c)

3. The first few terms in the Taylor series expansion of the function $f(x) = \sin x$ around $x = \frac{\pi}{4}$ are

[CSIR Dec 2011]

(a) $\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)+\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+.........\right]$

(b) $\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)+\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+.........\right]$

(c) $\left[\left(x-\frac{\pi}{4}\right)-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+.........\right]$

(d) $\frac{1}{\sqrt{2}}\left[1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+.......\right]$

Soln. **Taylor Series:** $f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)$

$f(x)=\sin x,\ f'(x)=\cos x,\ f''(x)=-\sin x,\quad f'''(x)=-\cos x$

$f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\ f'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\ f''\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$\therefore\quad f(x)=\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+......\right]$

Correct answer is (b)

4. The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $1 \le |z| \le 2$ and around $z = 1$ is

[CSIR June 2012]

(a) $\frac{1}{2}\left[1+z+z^2+....\right]\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+.......\right]$

(b) $\frac{1}{1-z}-z-(1-z)^2+(1-z)^3+........$

(c) $\frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+......\right]\left[1+\frac{2}{z}+\frac{4}{z^2}+........\right]$

(d) $2(z-1)+5(z-1)^2+7(z-1)^3+......$

Soln. $\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{1-z}+\frac{1}{z-2}=\frac{1}{1-z}+\frac{1}{(z-1)-1}$

$=\frac{1}{1-z}-\left[1-(z-1)\right]^{-1}=\frac{1}{1-z}-\left[1+(z-1)+(z-1)^2+......\right]$

$=\frac{1}{1-z}-z-(1-z)^2+(1-z)^3........$

Correct answer is (b).

5. Let $u(x,y)=x+\frac{1}{2}\left(x^2-y^2\right)$ be the real part of an analytic function $f(z)$ of the complex variable $z=x+iy$. the imaginary part of $f(z)$ is

(a) $y+xy$ (b) xy (c) y (d) y^2-x^2

[CSIR June 2012]

Soln. $u(x,y)=x+\frac{1}{2}\left(x^2-y^2\right);\ f(z)=u(x,y)+iv(x,y)$

Cauchy Reamann equation $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\frac{\partial u}{\partial x}=1+x \Rightarrow \frac{\partial v}{\partial y}=1+x$

$\Rightarrow \quad v=y+xy+g(x)$ ... (1)

$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \Rightarrow \frac{\partial u}{\partial y}=-y \Rightarrow \frac{\partial v}{\partial x}=y \Rightarrow v=xy+h(y)$ ... (2)

Compare (1) & (2), taking $h(y)=y$; and $g(x)=0$; $v=y+xy$

Correct answer is (a).

6. The value of the integral $\int_{-\infty}^{\infty}\frac{1}{t^2-R^2}\cos\left(\frac{\pi t}{2R}\right)dt$

[CSIR June 2012]

(a) $-\frac{2\pi}{R}$ (b) $-\frac{\pi}{R}$ (c) $\frac{\pi}{R}$ (d) $\frac{2\pi}{R}$

Soln. $\int_{-\infty}^{+\infty}\frac{1}{t^2-R^2}\cos\frac{\pi t}{2R}\,dt=$ Real part $\int_{-\infty}^{\infty}\frac{1}{t^2-R^2}\exp\left(\frac{\pi t}{2R}\right)dt$

$f(z)=\frac{1}{z^2-R^2}\exp\left(\frac{\pi z}{2R}\right)$ has two simple poles at $z=+R,-R$

Residue at $(z=R)=\frac{1}{2R}e^{\frac{i\pi}{2}}$; Residue at $(z=-R)=-\frac{1}{2R}e^{\frac{-i\pi}{2}}$

Since the two poles are on real axis, therefore, required integral = Real part ($\pi i \times$ sum of the residues)

$\Rightarrow I=$ Real part $\left[\pi i\left(\frac{1}{2R}\times e^{i\frac{\pi}{2}}-\frac{1}{2R}\times e^{-i\frac{\pi}{2}}\right)\right]=-\frac{\pi}{R}$

Correct option is (b).

7. The taylor series expansion of the function $\ln(\cosh x)$, where x is real, about point $x = 0$ starts with the following terms:

[CSIR Dec 2012]

(a) $-\frac{1}{2}x^2+\frac{1}{12}x^4+........$

(b) $\frac{1}{2}x^2-\frac{1}{12}x^4+........$

(c) $-\frac{1}{2}x^2+\frac{1}{6}x^4+........$

(d) $\frac{1}{2}x^2+\frac{1}{6}x^4+........$

Soln. Taylor series expansion of $f(x)$ about $x=x_0$ is:

$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)+...............$

Now, in question $x_0 = 0, f(x) = \ell n(\cosh x)$

$$f'(x) = \frac{1}{\cosh x}\sinh x = \tanh x$$

$$f''(x) = \sec h^2 x;$$

$$f'''(x) = 2\text{sech}\, x(-\text{sech}\, x.\tanh x) = -2\text{sech}^2 x.\tanh x$$

$$f''''(x) = -2(2\text{sech}\, x)(-\text{sech}\, x.\tanh x)\tanh x - 2\text{sech}^2 x.\text{sech}^2 x = 4\text{sech}^2 x.\tanh^2 x - 2\text{sech}^4 x$$

Required Taylor series expansion: $\ell n(\cosh x) = 0 + 0 + \frac{1}{2}.1.x^2 + 0 + \frac{1}{4!}(-2)x^4 + \ldots\ldots\ldots\ldots$

$$\Rightarrow \quad \ell n(\cosh x) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots\ldots\ldots$$

Correct answer is (b)

8. The value of the integral $\oint_C \frac{z^3}{z^2 - 5z + 6}dz$, where C is closed contour defined by the equation $2|z| - 5 = 0$, traversed in the anti-clockwise direction, is **[CSIR Dec 2012]**

(a) $-16\pi i$ (b) $16\pi i$ (c) $8\pi i$ (d) $2\pi i$

Soln. $f(z) = \dfrac{z^3}{z^2 - 5z + 6}$

Poles: $z^2 - 5z + 6 = 0$ Given closed contour: $c: 2|z| - 5 = 0 \Rightarrow |z| = \frac{5}{2}$

$\Rightarrow \quad (z-3)(z-2) = 0$

So, $z = 3 \Rightarrow$ point $(3, 0)$

$z = 2 \Rightarrow$ point $(2, 0)$

Only $z = 2$ i.e. point (2, 0) is within 'c'

centre (0, 0), radius 5/2 units

$$\text{Res.}f(z)\big|_{z=2} = (z-2)\frac{z^3}{(z-3)(z-2)}\bigg|_{z=2} = -8$$

Given integral: $I = 2\pi i \times (\text{sum of the residues}) = -16\pi i$

Correct answer is (a)

9. With $z = x + iy$, which of the following functions f (x,y) is NOT a complex analytic function of z? **[CSIR June 2013]**

(a) $(x+iy-8)^3(4+x^2-y^2+2ixy)^7$ (b) $(x+iy)^7(1-x-iy)^3$

(c) $(x^2-y^2+2ixy-3)^5$ (d) $(1-x+iy)^4(2+x+iy)^6$

Soln. Any complex function f(x,y) will be not analytic in nature if it contains $\overline{z}$ term. Given options can be written as

(a) $f(x,y) = (z-8)^3(4+z^2)^7$, (b) $f(x,y) = z^7(1-z)^3$

(c) $f(x,y) = (z^2-3)^5$, (d) $f(x,y) = (1-\overline{z})^4(2+z)$

Only option (d) contains $\overline{z}$ term. Therefore, option (d) will not be analytic in nature.

Correct answer is (d)

10. Which of the following functions cannot be the real part of a complex analytic function of $z = x + iy$? **[CSIR Dec 2013]**

(a) x^2y (b) $x^2 - y^2$ (c) $x^3 - 3xy^2$ (d) $3x^2y - y - y^3$

Soln. The real part and imaginary part of a complex analytic function of $z = x + iy$, will be harmonic in nature i.e. both will be satisfy Laplace's equation. Suppose, 'u' be the real part, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

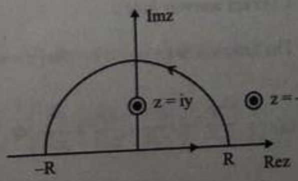
$$\text{Since, } \frac{\partial^2(x^2y)}{\partial x^2} + \frac{\partial^2(x^2y)}{\partial y^2} = 2y \neq 0$$

Correct answer is (a)

11. Given that the integral $\int_0^\infty \frac{dx}{y^2 + x^2} = \frac{\pi}{2y}$, the value of $\int_0^\infty \frac{dx}{(y^2+x^2)^2}$ is **[CSIR Dec 2013]**

(a) $\frac{\pi}{y^3}$ (b) $\frac{\pi}{4y^3}$ (c) $\frac{\pi}{8y^3}$ (d) $\frac{\pi}{2y^3}$

Soln. $\displaystyle\int_0^\infty \frac{dx}{(y^2+x^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(y^2+x^2)^2}$



So, $f(z) = \dfrac{1}{(z^2+y^2)^2}$ has poles of order '2' at $z \pm iy$

Since, both poles are not on real axis, then given integral

$$I = \frac{1}{2}\left[2\pi i(\text{Residue of } f(z) \text{ at } z = iy)\right]$$

$$= \frac{\pi i}{(2-1)!}\left[\frac{d}{dz}\left\{(z-iy)^2\frac{1}{(z^2+y^2)^2}\right\}\right]_{z=iy} = \pi i\left[-\frac{2}{(z+iy)^3}\right]_{z=iy} = \frac{\pi}{4y^3}$$

Correct answer is (b)

12. If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral **[CSIR June 2014]**

$$\oint_C \frac{dz}{\sin^2 z}$$

is

(a) ∞ (b) $2\pi i$ (c) 0 (d) πi

Soln. $f(z) = \frac{1}{\sin^2 z}$ has poles of order 2 at $z = n\pi \; (n = 0, 1, 2, \ldots\ldots)$

Only $z = 0$ will be within the circle $|z| = \frac{1}{2}$

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)^2} = \frac{1}{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right]^2}$$

$$= \frac{1}{z^2}\left[1 - \left\{\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right\}\right]^{-2}$$

$$= \frac{1}{z^2}\left[1 + 2\left\{\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right\} + 3\left\{\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right\}^2 + \ldots\right]$$

Residue of f(z) at $(z = 0)$ = coefficent of $\left(\frac{1}{z}\right)$ in the Laurent series expansion of f(z) about $(z = 0) = 0$

Therefore, the value of the integral = 0

Correct answer is (c).

13. The function $\Phi(x, y, z, t) = \cos(z - vt) + \text{Re}(\sin(x + iy))$ satisfies the equation **[CSIR June 2014]**

(a) $\frac{1}{v^2}\frac{\partial^2 \Phi}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi$ (b) $\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2}\right)\Phi$

(c) $\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\Phi = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\Phi$ (d) $\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\Phi = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\Phi$

Soln. $\phi(x, y, z, t) = \cos(z - vt) + \text{Re}(\sin(x + iy))$

As $\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \cosh y$

$\Rightarrow \phi(x, y, z, t) = \cos(z - vt) + \sin x \cosh y$

We get, $\frac{\partial^2 \phi}{\partial x^2} = -\sin x \cosh y$; $\frac{\partial^2 \phi}{\partial y^2} = \sin x \cosh y$; $\frac{\partial^2 \phi}{\partial z^2} = -\cos(z - vt)$; $\frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} = -\cos(z - vt)$

Therefore, $\frac{1}{v^2}\frac{\partial^2 \Phi}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi$

Correct answer is (a)

14. The principal value of the integral $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ is

(a) -2π (b) $-\pi$ (c) π (d) 2π

Soln. Given integral: $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ = Imaginary part of $\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^3} dx$

Assume, the complex function to be $f(z) = \frac{e^{i2z}}{z^3}$

Singular points of $f(z)$: $z = 0$ is pole of order 3
Since, the pole lie on the real axis, therefore,

$$\int_{-\infty}^{\infty} \frac{e^{i2z}}{z^3} dz = \pi i\left[\text{Residue of } f(z) \text{ at } z = 0\right]$$

$$= \pi i\left[\frac{1}{(3-1)!}\frac{d^2}{dz^2}\left\{(z-0)^3 \frac{e^{i2z}}{z^3}\right\}\right]_{z=0}$$

$$= \pi i\left[\frac{1}{2}(4i^2)e^{i2z}\right]_{z=0} = -2\pi i$$

So, the required integral $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ = Imaginary part of $(-2\pi i) = -2\pi$

Correct Option is (a)

15. The Laurent series expansion of the function $f(z) = e^z + e^{1/z}$ about $z = 0$ is given by

(a) $\sum_{n=-\infty}^{\infty} \frac{z^n}{n!}$ for all $|z| < \infty$ (b) $\sum_{n=0}^{\infty}\left(z^n + \frac{1}{z^n}\right)\frac{1}{n!}$ only if $0 < |x| < 1$

(c) $\sum_{n=0}^{\infty}\left(z^n + \frac{1}{z^n}\right)\frac{1}{n!}$ for all $0 < |z| < \infty$ (d) $\sum_{n=-\infty}^{\infty} \frac{z^n}{n!}$, only if $|z| < 1$

Soln. Given the complex function: $f(z) = e^z + e^{1/z}$

$$= \left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right] + \left[1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \ldots\right]$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{z^n}\frac{1}{n!} = \sum_{n=0}^{\infty}\left(z^n + \frac{1}{z^n}\right)\frac{1}{n!}$$

ince, e^z is not analytic at $z = \infty$ and $e^{1/z}$ is not analytic at $z = 0$,
Therefore, the laurent series expansion will be convergent for $0 < |z| < \infty$

Correct Option is (c)

16. Consider the function $f(z)=\frac{1}{z}\ln(1-z)$ of a complex variable $z=re^{i\theta}(r\geq 0,-\infty<\theta<\infty)$. The singularities of $f(z)$ are as follows:

(a) branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ only for $0\leq\theta<2\pi$
(b) branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ for all θ other than $0\leq\theta<2\pi$
(c) branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ for all θ
(d) branch points at $z=0$, $z=1$ and $z=\infty$

Soln. Given Complex function: $f(z)=\frac{1}{z}\ln(1-z)$

$$\Rightarrow f'(z)=-\frac{1}{z^2}\ln(1-z)+\frac{1}{z(1-z)}$$

Therefore, $f(z)$ is not differentiable at $z=0,1,\infty$.

The points $z=1,\infty$ comes from $\ln(1-z)$ and since $\ln(1-z)$ is a infinite valued function of z, then $z=1,\infty$ are two branch points of $f(z)$.

On the other hand, $z=0$ comes from $\frac{1}{z}$ i.e $z=0$ is a simple pole of $f(z)$. So, the residue of the function $f(z)$ at $z=0$ is

$$\text{Res.}\left[f(z=0)\right]=\underset{z\to 0}{Lt}(z-0)\frac{1}{z}\ln(1-z)=\underset{z\to 0}{Lt}\ln(1-z)$$

$$=\underset{r\to 1}{Lt}\ln\left[re^{i\theta}\right]=\underset{r\to 1}{Lt}\ln\left[re^{i(\theta+2k\pi)}\right]\quad(k=0,1,2,3,.....)$$

$$=\underset{r\to 0}{Lt}\left[\ln r+i(\theta+2k\pi)\right]$$

For the principal branch ($k=0$) between $0\leq\theta<2\pi$, if we take $\theta=0$, then the residue of the function $f(z)$ at $z=0$ will be zero. But for the other branches of the function $f(z)$, the residue will be non-zero. So, $z=0$ will be a pole of $f(z)$ for all θ other than $0\leq\theta<2\pi$.

Correct Option is (b)

GATE Previous Years Questions with Solutions

1. The value of the integral $\int_C z^{10}dz$, C is the unit circle with the origin as the centre is **[GATE 2001]**

(a) 0 (b) $z^{11}/11$ (c) $2\pi iz^{11}/11$ (d) 1/11

Soln. $f(z)=z^{10}$ is analytic in the complex argand plane.

So, according to Cauchy integral theorem,

$$\int_C z^{10}dz=0$$

Correct option is (a)

2. The value of the residue of $\frac{\sin z}{z^6}$ is **[GATE 2001]**

(a) $-\frac{1}{5!}$ (b) $\frac{1}{5!}$ (c) $\frac{2\pi i}{5!}$ (d) $-\frac{2\pi i}{5!}$

Soln. $$f(z)=\frac{\sin z}{z^6}=\frac{1}{z^6}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+.......\right]$$

$$=\frac{1}{z^5}-\frac{1}{3!}\frac{1}{z^3}+\frac{1}{5!}\frac{1}{z}-......$$

Residue of $f(z)$ = coefficient of $\left(\frac{1}{z}\right)=\frac{1}{5!}$

Correct option is (b)

3. If a function $f(z)=u(x,y)+iv(x,y)$ of the complex variable $z=x+iy$, where x, y, u and v are real analytic in a domain D of z, then which of the following is true? **[GATE 2002]**

(a) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}$
(b) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$
(c) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial y}$
(d) $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 v}{\partial x\partial y}$

Soln. A complex function $f(z)=u(x,y)+iv(x,y)$ will be analytic in nature if its real part and imaginary satisfies Cauchy-Reamann equation i.e.

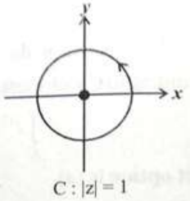
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\text{ and }\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Correct option is (b)

4. The value of the integral $\oint_C dz/z^2$, where z is a complex variable and C is the unit circle with the origin as its centre, is: **[GATE 2003]**

(a) 0 (b) $2\pi i$ (c) $4\pi i$ (d) $-4\pi i$

Soln. $f(z)=\frac{1}{z^2}$ has a pole of order 2 at $z=0$



C : |z| = 1

Therefore, $\int_C \frac{dz}{z^2}=2\pi i$ [Residue at $z=0$]

$$=2\pi i\left[\frac{1}{(2-1)!}\frac{d}{dz}\left\{z^2\cdot\frac{1}{z^2}\right\}_{z=0}\right]=0$$

Correct option is (a)

5. The inverse of the complex number $\frac{3+4i}{3-4i}$ is **[GATE 2004]**

(a) $\frac{7}{25}+i\frac{24}{25}$ (b) $-\frac{7}{25}+i\frac{24}{25}$ (c) $\frac{7}{25}-i\frac{24}{25}$ (d) $-\frac{7}{25}-i\frac{24}{25}$

Soln. $\left(\frac{3+4i}{3-4i}\right)^{-1}=\frac{3-4i}{3+4i}=\frac{(3-4i)^2}{9-16i^2}=\frac{-7-24i}{25}=-\frac{7}{25}-i\frac{24}{25}$

Correct option is (d)

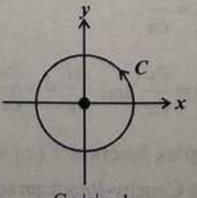
6. The value of $\int_C \frac{dz}{(z^2+a^2)}$, where C is a unit circle (anti clockwise) centered at the origin in the complex z-plane is **[GATE 2004]**

(a) π for $a=2$ (b) zero for $a=\frac{1}{2}$ (c) 4π for $a=2$ (d) $\frac{\pi}{2}$ for $a=\frac{1}{2}$

Soln. $f(z)=\frac{1}{z^2+a^2}$ has simple poles at $z=\pm ia$

For $a=2$, poles at $z=\pm 2i$ lies outside the circle $|z|=1$. So,

$$\oint_C \frac{dz}{z^2+a^2}=0$$



C : |z| = 1

For $a=\frac{1}{2}$, poles at $z=\pm\frac{i}{2}$ lies within the circle

$$\oint_C \frac{dz}{z^2+a^2}=2\pi i\left[\text{Res.}f\left(z=\frac{i}{2}\right)+\text{Res. }f(z)=-\frac{i}{2}\right]$$

$$=2\pi i\left[\left.\frac{1}{2z}\right|_{z=i/2}+\left.\frac{1}{2z}\right|_{z=-\frac{i}{2}}\right]=0$$

Correct option is (b)

7. The value of integral $\int_C \frac{dz}{z+3}$ where C is a circle (anticlockwise) with $|z|=4$, is **[GATE 2005]**

(a) 0 (b) πi (c) $2\pi i$ (d) $4\pi i$

Soln. $f(z)=\frac{1}{z+3}$ has a simple pole at $z=-3$

$$\int_C \frac{dz}{z+3}=2\pi i\left[\text{Res. }f(z=-3)\right]=2\pi i\left[(z+3)\frac{1}{(z+3)}\right]=2\pi i$$

Correct option is (c)

8. All solutions of the equations $e^z=-3$ are **[GATE 2005]**

(a) $in\pi\ln 3, n=\pm1,\pm2,\ldots\ldots\ldots$ (b) $ln3+i(2n+1)\pi, n=0,\pm1,\pm2,\ldots\ldots$

(c) $\ln 3+i2n\pi, n=0,\pm1,\pm2\ldots\ldots$ (d) $i3n\pi, n=\pm1,\pm2\ldots\ldots\ldots$

Soln. $e^z=-3\Rightarrow e^z=e^{i(2n+1)\pi}\cdot 3$

$$\Rightarrow\quad z=\ln 3+i(2n+1)\pi\qquad (n=0,\pm1,\pm2,\ldots\ldots)$$

Correct option is (b)

9. Consider the following function: $f(z)=\frac{\sin z}{z}$. Which of the following statemens is are TRUE? **[GATE 2005]**

(a) z = 0 is pole of order 1 (b) z = 0 is a removable singular point
(c) z = 0 is a pole order 3 (d) z = 0 is an essential singular point

Soln. z = 0 is a removable singular point of $f(z)=\frac{\sin z}{z}$ as

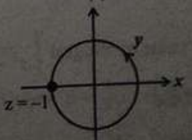
$$\frac{\sin z}{z}=\frac{1}{z}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}-+\ldots.\right]=1-\frac{z^2}{3!}+\frac{z^4}{5!}+\ldots.$$

Correct option is (b)

10. The value of $\int_C \frac{e^{2z}}{(z+1)^4}dz$, where C is a circle defined by $|Z|=3$, is **[GATE 2006]**

(a) $\frac{8\pi i}{3}e^2$ (b) $\frac{8\pi i}{3}e^{-1}$ (c) $\frac{8\pi i}{3}e$ (d) $\frac{8\pi i}{3}e^2$

C : |z| = 1



Soln. $f(z)=\frac{e^{2z}}{(z+1)^4}$ has a pole of order 4 at z = −1.

Residue of $f(z)$ at z = −1

$$=\frac{1}{(4-1)!}\frac{d^3}{dz^3}\left[(z+1)^4\frac{e^{2z}}{(z+1)^4}\right]_{z=-1}$$

$$=\frac{1}{3!}\left[8e^{2z}\right]_{z=-1}=\frac{4}{3}e^{-2}$$

So, $I = 2\pi i\left(\frac{4}{3}e^{-2}\right) = \frac{8\pi i}{3}e^{-2}$

Correct option is (a)

11. The contour integral $\oint \frac{dz}{z^2+a^2}$ is to be evaluated on a circle of radius 2a centered at the origin. It will have contributions only from the points. **[GATE 2007]**

(a) $\frac{1+i}{\sqrt{2}}a$ and $-\frac{1+i}{\sqrt{2}}a$ (b) ia and $-ia$

(c) $ia, -ia, \frac{1-i}{\sqrt{2}}a$ and $-\frac{1-i}{\sqrt{2}}a$ (d) $\frac{1+i}{\sqrt{2}}a, -\frac{1+i}{\sqrt{2}}a, \frac{1-i}{\sqrt{2}}a$ and $-\frac{1-i}{\sqrt{2}}a$

Soln. $f(z) = \frac{1}{z^2+a^2}$ has simple poles at $z = \pm ia$

Correct option is (b)

12. If $I = \oint_C \ln z dz$, where C is the unit circle taken anticlockwise and lnz is the principal branch of the logarithmic function, which of the following is correct? **[GATE 2008]**

(a) $I = 0$ by residue theorem. (b) I is not defined since, lnz is branch cut

(c) $I \neq 0$ (d) $\oint_C \ln(z^2)dz = 2I$

Soln. $f(z) = \ln z$ has a branch point at $z = 0$

Therefore, $\oint_C \ln(z)dz = I \neq 0$

$$\oint_C \ln(z^2)dz = 2\oint_C \ln(z)dz = 2I$$

13. The value of $\int_{-i}^{i} \pi(z+1)dz$ is **[GATE 2008]**

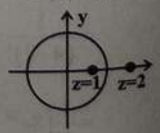
(a) 0 (b) $2\pi i$ (c) $-2\pi i$ (d) $(-1+2i)\pi$

Soln. $\int_{-i}^{i} \pi(z+1)dz = \pi\left[\frac{z^2}{2}+z\right]_{-i}^{i} = \pi\left[\frac{i^2}{2}+i-\frac{i^2}{2}+i\right] = 2\pi i$

Correct option is (b)

14. The value of integral $\int_C \frac{e^z}{z^2-3z+2}dz$, where the contour C is the circle $|z| = \frac{3}{2}$ is **[GATE 2009]**

(a) $2\pi ie$ (b) πie (c) $-2\pi ie$ (d) $-\pi ie$



Soln. $f(z) = \frac{e^z}{z^2-3z+2}$ has simple poles at $z = 1$ and $z = 2$

Only $z = 1$ lies within the circle.

$I = 2\pi i\left[\text{Residual of } f(z) \text{ at } z = 1\right]$

$= 2\pi\left[\lim_{z\to 1}(z-1)\frac{e^z}{(z-1)(z-2)}\right] = 2\pi i(-e) = -2\pi ie$

Correct option is (c)

15. For the complex function, $f(z) = \frac{e^{\sqrt{z}}-e^{-\sqrt{z}}}{\sin(\sqrt{z})}$, which of the following statement is correct?

(a) z = 0 is a branch point
(b) z = 0 is a pole of order one **[GATE 2010]**
(c) z = 0 is a removable singularity
(d) z = 0 is an essential singularity

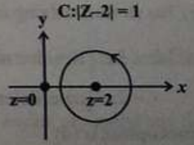
Soln. $f(z) = \frac{e^{\sqrt{z}}-e^{-\sqrt{z}}}{\sin(\sqrt{z})} = 2\frac{\sinh(\sqrt{z})}{\sin(\sqrt{z})}$ is not analytic at $z = 0$

$$\lim_{z\to 0} 2\cdot\frac{\sinh(\sqrt{z})}{\sin(\sqrt{z})} = 2\lim_{z\to 0}\frac{\cosh(\sqrt{z})\cdot\frac{1}{2\sqrt{z}}}{\cos(\sqrt{z})\cdot\frac{1}{2\sqrt{z}}} = 2$$

So, $z = 0$ is a removable singular point.
Correct option is (c)

16. The value of integral $\oint_C \frac{e^z \sin z}{z^2}dz$, where the contour C is the unit circle : $|z-2| = 1$, is

(a) $2\pi i$ (b) $4\pi i$ (c) πi (d) 0 **[GATE 2010]**



Soln. $f(z) = \frac{e^z \sin z}{z^2}$ has pole of order '2' at z = 0

Since, $z = 0$ lies outside the circle

So, $\oint_C \frac{e^z\cdot\sin z}{z^2}dz = 0$

Correct option is (d)

17. Which of the following statements is **TRUE** for the function $f(z)$?

(a) $f(z)$ is analytic everywhere in the complex plane

(b) $f(z)$ has a zero at $z = \pi$

(c) $f(z)$ has a pole of order 2 at $z = \pi$

(d) $f(z)$ has a simple pole at $z = \pi$

18. Consider a counterclockwise circular contour $|z| = 1$ about the origin. The integral $\oint f(z)dz$ over this contour is :

(a) $-i\pi$ (b) zero (c) $i\pi$ (d) $2i\pi$

Soln. $f(z) = \frac{z \sin z}{(z-\pi)^2}$ putting $z - \pi = t$

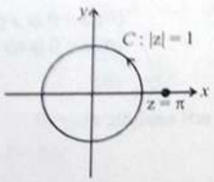
$$= \frac{(t+\pi)\sin(t+\pi)}{t^2} = \frac{(t+\pi)[-\sin t]}{t^2}$$

$$\Rightarrow f(z) = -\frac{\sin t}{t} - \pi\frac{\sin t}{t^2} = \frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\ldots\right] - \pi\cdot\frac{1}{t^2}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\ldots\right]$$

So, $f(z)$ has a simple pole at $z = \pi$

Correct option is (d)

Soln. Since, $z = \pi$ lies outside the circle.



Therefore, $\oint_C \frac{z\sin z}{(z-\pi)^2}dz = 0$

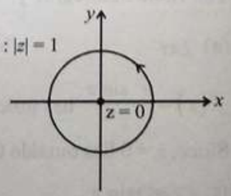
Correct option is (b)

19. The value of the integral $\oint_C e^{1/z}dz$, using the contour C of circle with unit radius $|z| = 1$, is **[GATE-2012]**

(a) 0 (b) $1 - 2\pi i$ (c) $1 + 2\pi i$ (d) $2\pi i$

Soln. $f(z) = e^{1/z} = \left[1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \ldots\ldots\right]$

$f(z)$ has an essential singular point at $z = 0$.



$$\oint_C e^{-1/z}dz = 2\pi i\,[\text{Residue of } f(z) \text{ at } z = 0]$$
$$= 2\pi i\,[\text{co-efficients of } 1/z] = 2\pi i$$

Correct option is (d)

20. For the function $f(z) = \frac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z = 1$......... **[GATE-2013]**

(Your answer should be an integer)

Soln. $f(z) = \frac{16z}{(z+3)(z-1)^2}$ has a pole of order 2 at $z = 1$

Residue of $f(z)$ at $z = 1$

$$= \frac{1}{(2-1)!}\frac{d}{dz}\left[(z-1)^2\frac{16z}{(z+3)(z-1)^2}\right]_{z=1} = \left.\frac{(z+3)\cdot 16 - 16z\cdot 1}{(z+3)^2}\right|_{z=1} = \left.\frac{48}{(z+3)^2}\right|_{z=1} = 3$$

Correct answer is (3)

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21. The value of the integral $\oint_C \frac{z^2}{e^z+1}dz$, where C is the circle $|z| = 4$, is **[GATE-2014]**

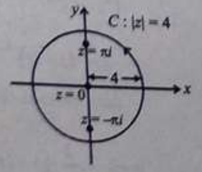
(a) $2\pi i$ (b) $2\pi^2 i$ (c) $4\pi^3 i$ (d) $4\pi^2 i$

Soln. $f(z) = \frac{z^2}{e^z+1}$

Condition of singularity $e^z + 1 = 0$

$\Rightarrow e^z = -1 = e^{\pm(2n+1)\pi i}$

$\Rightarrow z = \pm i(2n+1)\pi = \pm i\pi, \pm i3\pi, \pm i5\pi\ldots\ldots\ldots$



Only $z = \pm\pi i$ lies within the circle and it is a simple pole.

$I = 2\pi i$ [Residue of $f(z)$ at $z = \pi i$ + Residue of $f(z)$ at $z = -\pi i$]

$$= 2\pi i\left[\frac{z^2}{e^z}\right]_{z=\pi i} + 2\pi i\left[\frac{z^2}{e^z}\right]_{z=-\pi i} = 4\pi^3 i$$

Correct option is (c)

22. Consider $w = f(z) = u(x,y) + iv(x,y)$ to be an analytic function in a domain D. Which one of the following options is NOT correct ? **[GATE 2015]**

(a) $u(x,y)$ satisfies Laplace equation in D

(b) $v(x,y)$ satisfies Laplace equation in D

(c) $\int_{z_1}^{z_2} f(z)dz$ is dependent on the choice of the contour between z_1 and z_2 in D

(d) $f(z)$ can be Taylor expanded in D

Soln. $f(z) = u(x,y) + iv(x,y)$

The real part $u(x,y)$ and imaginary part $v(x,y)$ of a complex analytic function $f(z)$ are harmonic functions i.e. they will satisfy Laplaces equation.

Since, $f(z)$ is analytic in the domain D, then $f(z)$ can be taylor expanded and the value of $\int_{z_1}^{z_2} f(z)dz$ will be independent of the choice of the contour.

Correct option is (c)

23. Consider a complex function $f(z) = \frac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$. Which one of the following statements is correct ? **[GATE 2015]**

(a) $f(z)$ has simple poles at $z = 0$ and $z = -\frac{1}{2}$

(b) $f(z)$ has a second order pole at $z = -\frac{1}{2}$

(c) $f(z)$ has infinite number of second order poles

(d) $f(z)$ has all simple poles

Soln. $f(z)=\dfrac{1}{z\left(z+\dfrac{1}{2}\right)\cos(z\pi)}$ putting $z+\dfrac{1}{2}=t$

$$=\frac{1}{\left(t-\dfrac{1}{2}\right)t\cos\left[\pi\left(t-\dfrac{1}{2}\right)\right]}=\frac{1}{\left(t-\dfrac{1}{2}\right)t\sin\pi t}=\frac{1}{-\dfrac{1}{2}(1-2t)}\cdot\frac{1}{t}\cdot\frac{1}{\left[\pi t-\dfrac{\pi^3t^3}{3!}+.........\right]}$$

$$=-2\cdot\frac{1}{t}\cdot(1-2t)^{-1}\cdot\frac{1}{\pi t\left[1-\dfrac{\pi^2t^3}{3!}+......\right]}$$

$$=-2\cdot\frac{1}{t}\cdot\frac{1}{\pi t}\cdot\left[1+2t+4t^2+......\right]\left[1+\left(\frac{\pi^2t^2}{3!}+.....\right)+\left(\frac{\pi^2t^2}{3!}+.....\right)^2+......\right]$$

Highest negative power term in above Laurrent series expansion is $\dfrac{1}{t^2}$

Therefore, $f(z)$ has a pole of order 2 at $z=-\dfrac{1}{2}$

Correct option is (b)

PRACTICE SET

1. If $x+iy=\left(1-i\sqrt{3}\right)^{100}$, then the co-ordinate of point P(x, y) =

(a) $\left(2^{99},2^{99}\sqrt{3}\right)$ (b) $\left(2^{99},-2^{99}\sqrt{3}\right)$ (c) $\left(-2^{99},2^{99}\sqrt{3}\right)$ (d) none of these

2. The smallest positive integer n for which $\left(\dfrac{1+i}{1-i}\right)^n$ is real

(a) 3 (b) 2 (c) 4 (d) none

3. The sum of the following series:

$$i^2+i^4+i^6....\text{ up to }(2n+1)\text{ terms}$$

will be

(a) i (b) $-i$ (c) 1 (d) – 1

4. If 1, ω, ω^2 are the cube roots of unity, then $\left(1-\omega+\omega^2\right)^5+\left(1+\omega-\omega^2\right)^5$ will be

(a) 4 (b) 8 (c) 16 (d) 32

5. If 1, ω, ω^2 be cube roots of unity, then the sum of the following series:

$$\left(1-\omega+\omega^2\right)\left(1-\omega^2+\omega^4\right)\left(1-\omega^4+\omega^8\right)....\text{ to }2n\text{ factors}$$

will be

(a) $2n$ (b) 2^{2n} (c) 0 (d) 1

6. If sum and products of two complex numbers are real, then which of the following statements is/are true?

(a) both complex numbers are real (b) both complex numbers are purely imaginary
(c) one of the complex numbers are real (d) both are complex conjugate of each other

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7. The expression $\sinh^{-1}x$ can be simplified to

(a) $\ln\left(x+\sqrt{x^2+1}\right)$ (b) $\ln\left(x+\sqrt{x^2-1}\right)$ (c) $\ln\left(x-\sqrt{x^2-1}\right)$ (d) $\ln\left(x-\sqrt{x^2+1}\right)$

8. The value of the following limit:

$$\underset{z\to0}{Lt}\frac{\mathrm{Re}\left(z^2\right)+\mathrm{Im}\left(z^2\right)}{z^2}$$

(a) 0 (b) 1 (c) -1 (d) does not exist

9. The value of the following limit:

$$\underset{z\to\infty}{Lt}\frac{iz^3+iz-1}{(2z+3i)(z-i)^2}$$

(a) $\dfrac{i}{2}$ (b) $-\dfrac{i}{2}$ (c) $\dfrac{1}{2}$ (d) does not exist

10. Which of the foloowing statement is true for the compex function $f(z)=\dfrac{1}{z}$?

(a) $f(z)$ is analytic everywhere (b) $f(z)$ is analytic everywhere except z = 0

(c) $f(z)$ is analytic nowhere (d) $f(z)$ is analytic everywhere except z = 1

11. Which of the following are not a complex analytic function of z ?

(a) $f(z)=\left(x^2-y^2+2ixy\right)^7(x+iy)^{17}$ (b) $f(z)=\left(x^2-y^2+2ixy\right)^{12}(x-iy)^4$

(c) $f(z)=(x+iy-5)^{13}$ (d) $f(z)=(2x+iy-5)^{19}$

12. Given: $f(z)=x^2+Py^2-2xy+i\left(Qx^2-y^2+2xy\right)$ is analytic in nature. The value of P and Q will be

(a) P = 1, Q = 1, (b) P = -1, Q = 1, (c) P = 1, Q = -1, (d) P = -1, Q = -1

13. An electtrostatic field in the x-y plane is given by the potential function $\phi(x,y)=3x^2y-y^3$. The corresponding stream function will be

(a) $-x^3-3x^2y$ (b) $-x^3+3xy^2$ (c) $-x^3-3xy^2$ (d) x^3-3xy^2

14. The value of the integral $\displaystyle\int_0^{1+i}\left(x^2-iy\right)dz$ along the path $y=x^2$ is

(a) $\dfrac{5}{6}+\dfrac{i}{6}$ (b) $\dfrac{5}{6}-\dfrac{i}{6}$ (c) $\dfrac{1}{6}-\dfrac{5i}{6}$ (d) $\dfrac{1}{6}+\dfrac{5i}{6}$

15. The taylor series expansion of $f(z)=\cos z$ about $z=\frac{\pi}{6}$ will be

(a) $f(z)=\frac{\sqrt{3}}{2}+\frac{1}{2}\left(z-\frac{\pi}{6}\right)+\frac{\sqrt{3}}{4}\left(z-\frac{\pi}{6}\right)^2+\ldots\ldots\ldots$

(b) $f(z)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(z-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(z-\frac{\pi}{6}\right)^2+\ldots\ldots\ldots$

(c) $f(z)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(z-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{2}\left(z-\frac{\pi}{6}\right)^2+\ldots\ldots\ldots$

(d) $f(z)=\frac{\sqrt{3}}{2}+\frac{1}{2}\left(z-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{2}\left(z-\frac{\pi}{6}\right)^2-\ldots\ldots\ldots$

16. If $\sin^{-1}z$ is expanded in terms of z, it can be written as

(a) $z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots\ldots\ldots$
(b) $z-\frac{z^3}{6}+\frac{3z^5}{40}-\frac{5z^7}{112}+\ldots\ldots\ldots$
(c) $z+\frac{z^3}{3}+\frac{z^5}{5}+\frac{z^7}{7}+\ldots\ldots\ldots$
(d) $z+\frac{z^3}{6}+\frac{3z^5}{40}+\frac{5z^7}{112}+\ldots\ldots\ldots$

17. Suppose a complex function f(z) such that $f(1)=1, f'(1)=1, f''(1)=1$ and all other higher derivatives of $f(z)$ at $z=1$. The value of $f(z)$ at $z=1/2$ will be

(a) 1/4 (b) 1/2 (c) 5/8 (d) 7/8

18. In the Laurent series expansion of $f(z)=\frac{(z^2+3z+2)}{z(z+2)^3}$ about $z=-2$, the highest negative power (<0) terms of $(z+2)$ present in the expansion, will be

(a) 0 (b) 1 (c) 2 (d) 3

19. The Laurent series expansion of the complex function, $f(z)=\frac{z}{(z-1)(z-2)}$ in thr region $|z-1|>1$ can be expressed as

(a) $\frac{1}{z-1}+\frac{2}{(z-1)^2}+\frac{2}{(z-1)^3}+\frac{2}{(z-1)^4}+\ldots\ldots\ldots$

(b) $\frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{2}{(z-1)^3}-\frac{2}{(z-1)^4}+\ldots\ldots\ldots$

(c) $\frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\frac{4}{(z-1)^4}+\ldots\ldots\ldots$

(d) $\frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{1}{(z-1)^3}-\frac{2}{(z-1)^4}+\ldots\ldots\ldots$

20. The function $f(z)=(z-3)\sin\left(\frac{1}{z+2}\right)$ has singularity at $z=-2$ of the following nature :

(a) pole of order 1
(b) removable isolated singularity
(c) isolated essential singularity
(d) non-isolated essential singularity

21. The function $f(z)=\frac{1}{1-e^z}$ has singularity at $z=2\pi i$ of the following nature:

(a) pole of order 1
(b) removable isolated singularity
(c) isolated essential singularity
(d) non-isolated essential singularity

22. The function $f(z)=(z-3)^n\sin\left(\frac{1}{z-3}\right)$ has a residue of $\frac{1}{120}$ at the point $z=3$. The value of 'n' is equal to

(a) 2 (b) 4 (c) 5 (d) 6

23. The contribution of the point $z=\pi/2$ in evaluation of $\oint_C \frac{\tan z}{z}dz$ (where C is a circle $|z|=2$) is equal to

(a) 0 (b) $-4e^{i\pi/2}$ (c) $4e^{i\pi/2}$ (d) $-2/\pi$

24. The value of the integral $\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$, where C is a circle $|z|=2$ in the argand plane, described in the positive sense is equal to

(a) $\pi/2$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/5$

25. The value of the integral $\oint_C \frac{3z^2+z+1}{(z^2-1)(z+3)}dz$ around the curve $C:|z|=2$ (where, 'C' is traversed in the clockwise direction) is equal to

(a) $\frac{3\pi i}{4}$ (b) $-\frac{3\pi i}{4}$ (c) $\frac{\pi i}{4}$ (d) $-\frac{\pi i}{4}$

26. The value of the integral $\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}dz$, where 'C' is the circle $|z|=3$ traversed in the anticlockwise direction, will be

(a) $2\pi i$ (b) $-2\pi i$ (c) $4\pi i$ (d) $-4\pi i$

27. The value of the integral $\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta}$ will be

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{6}$

28. The value of the integral $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\,d\theta$ will be

(a) $\frac{2\pi}{(n+1)!}$ (b) $\frac{2\pi}{n!}$ (c) $\frac{\pi}{n!}$ (d) $\frac{\pi}{(n+1)!}$

29. The value of the integral $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$ will be

(a) $\frac{5\pi}{12}$ (b) $\frac{5\pi}{24}$ (c) $\frac{7\pi}{12}$ (d) $\frac{7\pi}{24}$

30. The value of the integral $\int_0^{\infty} \frac{\sin mx}{x\left(x^2+a^2\right)} dx$ will be

(a) $\frac{\pi}{2a^2}\left(1+e^{-ma}\right)$ (b) $\frac{\pi}{2a^2}\left(1-e^{-ma}\right)$ (c) $\frac{\pi}{2a^2}\left(1-e^{ma}\right)$ (d) $\frac{\pi}{2a^2}\left(1+e^{ma}\right)$

31. The following integral $\int_0^{\infty} \frac{\cos 3x}{\left(x^2+1\right)\left(x^2+4\right)} dx$ has a value

(a) $\frac{\pi}{2}\left[\frac{e^3}{3}-\frac{e^6}{6}\right]$ (b) $\frac{\pi}{2}\left[\frac{e^{-3}}{3}-\frac{e^{-6}}{6}\right]$ (c) $\pi\left[\frac{e^{-3}}{3}-\frac{e^{-6}}{6}\right]$ (d) $\pi\left[\frac{e^3}{3}-\frac{e^6}{6}\right]$

32. Check whether the following functions are analytic or not:

(i) $\frac{1}{z}$ (ii) $z|z|$ (iii) $e^x\left(\cos y+i\sin y\right)$ (iv) $e^{\sin z}$ (v) $r^n e^{in\theta}$

33. Given: $f(z)=x^2+Py^2-2xy+i\left(Qx^2-y^2+2xy\right)$ is analytic in nature. Find P, Q and $f'(z)$.

34. Determine the imaginary or real part of the complex analytic function whose real or imaginary part are as follows:

(i) $v(x,y)=6xy-5x+3$ (ii) $v(r,\theta)=r^2\cos 2\theta-r\cos\theta+2$ (iii) $u(r,\theta)=-r^3\sin 2\theta$

35. Find the complex analytic function $f(z)$ forwhich: (Use Milne Thomson Method)

(i) $u=e^{-2xy}\sin\left(x^2-y^2\right)$ (ii) $v=\tan^{-1}\left(\frac{y}{x}\right)$ (iii) $u=\log\sqrt{x^2+y^2}$

36. Show that the function $u=\sin x.\cosh y+2\cos x.\sinh y+x^2-y^2+4xy$ is harmonic in nature and determine the corresponding analytic function.

37. If $f(z)=u+iv$ is an analytic fuinction of $z=x+iy$ and $u-v=\frac{e^y-\cos x+\sin x}{\cosh y-\cos x}$, find f(z) subject to the condition $f\left(\frac{\pi}{2}\right)=\frac{3-i}{2}$.

38. Find the first three terms of the Taylor's series expansion of $f(z)=\frac{z}{(z-3)(z-4)}$ about $z=2$. Find the region of convergence.

39. Expand $\cos z$ in Taylor's series about $z=\frac{\pi}{6}$.

40. Expand the following functions into Laurent series:

(i) $f(z)=\frac{z^2-6z-1}{(z-1)(z+2)(z-3)}$ for the region $3<|z+2|<5$.

(ii) $f(z)=\frac{1}{z(z+2)^3}$ about the pole $z=-2$

41. Discuss the nature of singularity regarding the following complex functions:

(i) $f(z)=\frac{1}{1-e^z}$ at $z=2\pi i$ (ii) $f(z)=\frac{1}{\cos z-\sin z}$ (iii) $f(z)=\frac{\cot\pi z}{(z-a)^2}$ at $z=a, z=\infty$

(iv) $f(z)=\frac{1-e^z}{1+e^z}$ at $z=\infty$ (v) $f(z)=\frac{z}{1+z^4}$ (vi) $f(z)=\frac{\sin z}{(z-\pi)^2}$

42. Determine the poles and corresponding residues of following complex functions:

(i) $f(z)=\frac{z^2}{(z+1)^2(z-2)}$ (ii) $f(z)=\frac{z}{\sin z}$ (iii) $f(z)=\frac{e^z}{z^2(z^2+9)}$ at (0,-3)

(iv) $f(z)=\frac{2z+1}{z^2-z-2}$ (v) $f(z)=z^2e^{1/z}$ (vi) $f(z)=z^2\sin\left(\frac{1}{z}\right)$

43. Evaluate the following integral:

(i) $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$; C : circle of $|z-i|=2$ (ii) $\oint_C \frac{1}{\sinh z} dz$; C : circle of $|z|=4$

(iii) $\int_C z^4 e^{1/z} dz$; C : circle of $|z|=1$ (iv) $\int_C \frac{\sin z}{z^6} dz$; C : circle of $|z|=2$

(v) $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} d\theta$ (vi) $\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta} d\theta$

(vii) $\int_0^{2\pi} \frac{d\theta}{1-2m\cos\theta+m^2}$, where $m^2<1$ (viii) $\int_0^{2\pi} \frac{\sin^2\theta-2\cos\theta}{2+\cos\theta} d\theta$

(ix) $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$ (x) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\,d\theta$

(xi) $\int_0^{\infty} \frac{dx}{x^4+a^4}$ (xii) $\int_{-\infty}^{\infty} \frac{x^2}{\left(1+x^2\right)\left(x^2+4\right)} dx$

ANSWER KEY

1. (c)	**2. (b)**	**3. (d)**	**4. (d)**	**5. (b)**	**6. (a, d)**	**7. (a)**
8. (d)	**9. (a)**	**10. (b)**	**11. (b, d)**	**12. (b)**	**13. (b)**	**14. (a)**
15. (b)	**16. (d)**	**17. (c)**	**18. (c)**	**19. (a)**	**20. (c)**	**21. (a)**
22. (b)	**23. (a)**	**24. (d)**	**25. (c)**	**26. (c)**	**27. (a)**	**28. (b)**
29. (a)	**30. (b)**	**31. (b)**	**32. (c)**	**33. (b)**	**34. (b)**	**35. (b)**
36. ( )	**37. ( )**	**38. (a)**	**39. (d)**	**40. (a)**	**41. (b)**	

Chapter 6

FOURIER TRANSFORM

6.1 Infinite Fourier Transform:

The fourier transform of $f(x)$ $[-\infty < x < \infty]$:

$$f(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Inverse fourier transform of $f(s)[-\infty < s < \infty]$:

$$f(x) = F^{-1}\{f(s)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-isx}ds$$

6.2 Fourier Sine Transform:

Fourier sine transform of $f(x)$:

$$f_s(s) = F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx$$

Inverse fourier sine transform of $f(s)$:

$$f(x) = F_s^{-1}\{f_s(s)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(s)\sin sx\, ds$$

6.3 Fourier Cosine Transform

Fourier cosine transform of $f(x)$:

$$f_c(s) = F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$$

Inverse fourier cosine transform of $f(s)$:

$$f(x) = F_c^{-1}\{f_c(s)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_c(s)\cos sx\, ds$$

Important Formula:

(i) $\int_0^{\infty} \frac{\sin ax}{x}dx = \frac{\pi}{2}$ (ii) $\int_0^{\infty} e^{-x^2}dx = \sqrt{\frac{\pi}{2}}$ (iii) $\int_0^{\infty} e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}$

(iv) $\int_0^{\infty} e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}$ (v) $\int_0^{\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ (vi) $\int_0^{\infty} e^{-ax^2}\cos bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-b^2/4a}$

(vii) $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$

Example 1: Find the fourier transform of Gaussian distribution function $f(x) = Ne^{-ax^2}$ where N and 'a' are constant.

Soln: $F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Ne^{-ax^2}e^{isx}dx = \frac{N}{\sqrt{2\pi}}\sqrt{\frac{\pi}{a}}e^{-s^2/4a} = \frac{N}{\sqrt{2a}}e^{-s^2/4a}$

Example 2: Find the fourier transform of the function $f(x) = \delta(x)$

Soln: $F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\left[e^{isx}\right]_{x=0} = \frac{1}{\sqrt{2\pi}}$

Example 3: Find the fourier transform of $f(x) = e^{-a|x|}$ $(-\infty < x < \infty)$ $(a > 0)$

Soln: $F\left[e^{-a|x|}\right] = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ax}e^{isx}dx + \int_0^{\infty} e^{-ax}e^{isx}dx\right] = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{(a+is)x}}{(a+is)}\right)_{-\infty}^{0} + \left(\frac{e^{-(a-is)x}}{(is-a)}\right)_{0}^{\infty}\right]$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} + \frac{1}{a-is}\right] = \frac{2a}{\sqrt{2\pi}\left(s^2+a^2\right)}$$

Example 4: If $F\left\{e^{-i2\pi vat}\right\} = f(\omega+a)$, then $F(\cos 2\pi vat) = ?$

Soln: $F[\cos 2\pi vat] = F\frac{\left[e^{i2\pi vat} + e^{-i2\pi vat}\right]}{2} = \frac{1}{2}\left\{F\left[e^{i2\pi vat}\right] + F\left[e^{-i2\pi vat}\right]\right\} = \frac{1}{2}\left[f(\omega-a) + f(\omega+a)\right]$

Example 5: Find the fourier cosine transform of the function $f(x) = e^{-a^2x^2}$

Soln: $f_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-a^2x^2}\cos sx dx = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}\cos sx dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\text{ Real part of }\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx = \frac{1}{2}\sqrt{\frac{2}{\pi}}\text{ Real part of}\left(\sqrt{\frac{\pi}{a^2}}e^{\frac{i^2s^2}{4a^2}}\right) = \frac{1}{\sqrt{2a^2}}e^{-s^2/4a^2}$$

6.4 Properties of Fourier Transform

(1) Linearity theorem:

If $F(x)=a_1f_1(x)+a_2f_2(x)+\ldots.$ then, $f(s)=F\left[f(x)\right]=a_1f_1(s)+a_2f_2(s)+\ldots.$

(2) Change of scale property:

If $F\left[f(x)\right]=f(s)$, then $F\{f(ax)\}=\frac{1}{a}f(s/a)$

(3) Shifting property:

If $F\left[f(x)\right]=f(s)$, then $F\{f(x\pm a)\}=e^{\mp isa}f(s)$

(4) Modulation theorem:

If $F\{f(x)\}=f(s)$, then $F\{f(x)\cos ax\}=\frac{1}{2}f(s-a)+\frac{1}{2}f(s+a)$

(5) If $F\{f(x)\}=f(s)$, then $F\{f^*(x)\}=f^*(-s)$

(6) If $F_c\left[f(x)\right]=f_c(s)$, then $F_c\left[f\left(\frac{x}{a}\right)\right]=af_c(as)$

(7) If $F_c\left[f(x)\right]=f_c(s)$, then $F_c\left[f\left(\frac{x}{a}\right)\right]=af_c(as)$

(8) Parseval's Identity:

If the Fourier transform of $f(x)$ and $g(x)$ be $f(s)$ & $g(s)$ respectively, then

(i) $\int_{-\infty}^{\infty}f(s)g^*(s)\,ds=\int_{-\infty}^{\infty}f(x)g^*(x)\,dx$

where $g^*(s)$ is complex conjugate of $g(s)$ and $g^*(x)$ is the complex conjugate of $g(x)$.

(ii) $\int_{-\infty}^{\infty}\left|f(s)\right|^2ds=\int_{-\infty}^{\infty}\left|f(x)\right|^2dx$

(9) Derivative theorem:

If $F\{f(x)\}=f(s)$, then $F\left[\frac{df}{dx}\right]=-isf(s)$

If $F\{f(x)\}=f(s)$, then $F\left[\frac{d^nf}{dx^n}\right]=(-is)^nf(s)$

Example 6: (a) Find fourier transform of $f(x)=\begin{cases}1 & for\ |x|<a\\ 0 & for\ |x|>a\end{cases}$

(b) Using parseval's identity show that $\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2dx=\pi/2$

Soln: $F\{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx=\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$

Using parseval identity, $\int_{-\infty}^{\infty}\left|f(x)\right|^2dx=\int_{-\infty}^{\infty}\left|f(s)\right|^2ds$

$$\Rightarrow\quad \int_{-a}^{a}1^2dx=\int_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sin^2 as}{s^2}ds\quad\Rightarrow\quad 2a=\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2ds$$

Putting $as=x\Rightarrow ads=dx$

$$\Rightarrow\quad \pi=\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2dx\qquad\Rightarrow\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2dx=\frac{\pi}{2}$$

Example 7: If the fourier transform of $f(x)$ is $f(s)$, then find the fourier transform of $\frac{df}{dx}$

Soln: $F\left[\frac{df}{dx}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df}{dx}e^{isx}dx=\frac{1}{\sqrt{2\pi}}\left[e^{isx}f(x)\right]_{-\infty}^{\infty}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}is\,e^{isx}f(x)\,dx$

Since $f(x)$ should be a well behave function for the fourier tranform to exist, therefore, the first term of the above equation should be equal to zero.

Then $F\left[\frac{df}{dx}\right]=-is\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)\,dx=-isf(s)$

Example 8: Find Fourier transform of $f(x)=\begin{cases}x^2, & |x|<a\\ 0, & |x|>a\end{cases}$

Soln: $F\{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)\,dx=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{isx}x^2dx$

$$=\frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{isx}}{is}.x^2\right)_{x=-a}^{a}-\frac{2}{is}\int_{-a}^{a}xe^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{is}\left(e^{isa}-e^{-isa}\right)-\frac{2}{is}\left[\left(\frac{xe^{isx}}{is}\right)_{x=-a}^{a}-\frac{1}{is}\int_{-a}^{a}e^{isx}dx\right]\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{2a^2}{s}\sin(sa)+\frac{2a}{s^2}\cos(sa)-\frac{4}{s^3}\sin(sa)\right]$$

Example 9: Find the complex fourier transform $e^{-|x|}$

Soln: $F\left\{e^{-|x|}\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-|x|}e^{isx}dx=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{-|x|}e^{isx}dx+\int_{0}^{\infty}e^{-|x|}.e^{isx}dx\right]$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{x}.e^{isx}dx+\int_{0}^{\infty}e^{-x(1-is)}dx\right]=\frac{1}{\sqrt{2\pi}}\left[\left[\frac{e^{x(1+is)}}{1+is}\right]_{x=-\infty}^{0}+\left[\frac{e^{-x(1-is)}}{-(1-is)}\right]_{x=0}^{\infty}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-is}+\frac{1}{1+is}\right]=\frac{1}{\sqrt{2\pi}}\left(\frac{2}{1+s^2}\right)$$

Example 10: Find cosine transform of $f(x)$ if $f(x)=\begin{cases}\cos x, & 0<x<a\\ 0, & x>a\end{cases}$

Soln: $F_C\{f(x)\}=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos(px)\,dx$

$$=\sqrt{\frac{2}{\pi}}\int_0^{a}\cos x\cos(px)\,dx+\int_a^{\infty}0.\cos px dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^{a}\left[\cos(p+1)x\cos(p-1)x\right]dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\left[\frac{\sin(p+1)x}{p+1}+\frac{\sin(p-1)x}{p-1}\right]_{x=0}^{a}$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\left[\frac{\sin(p+1)a}{p+1}+\frac{\sin(p-1)a}{p-1}\right]$$

CSIR-NET/JRF Previous Years Questions with Solutions

1. Fourier transform of the derivative of the Dirac δ-function, namely $\delta'(x)$, is proportional to

(a) 0 (b) 1 (c) sin$k$ (d) ik

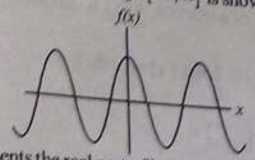
Soln. Fourier transform of $\delta'(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta'(x)e^{ikx}dx$

Using the property of Dirac delta function $\int_{-\infty}^{\infty}f(x)\delta'(x-a)\,dx=-f'(a)$

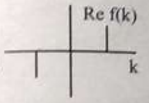
So, $F[\delta'(x)]=-\frac{1}{\sqrt{2\pi}}(ik)\quad\Rightarrow F[\delta'(x)]\propto ik$

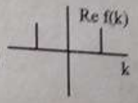
Correct answer is (b)

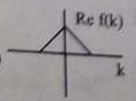
2. The graph of a real periodic function $f(x)$ for the range $[-\infty,\infty]$ is shown below

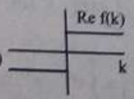


Which of the following graphs represents the real part of its Fourier transform?

(a)


(b)


(c)


(d)


Soln. The given graph in the question is of $f(x)=\cos\omega x$

$$F(\cos\omega x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\cos\omega x\, e^{ikx}dx=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty}\frac{e^{i(k+\omega)x}}{2}dx+\int_{-\infty}^{+\infty}\frac{e^{i(k-\omega)x}}{2}dx\right)$$

$$\Rightarrow F(\cos\omega x)\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}\delta(k+\omega)+\frac{1}{2}\delta(k-\omega)\right)$$

Therefore, the fourier transform of $f(x)=\cos x\omega x$ consist of two delta functions of same height at $k=\omega$ and $k=-\omega$.

Correct answer is (b)

GATE Previous Years Questions with Solutions

1. Fourier transform of which of the following functions does not exist? [GATE 2002]

(a) $e^{-|x|}$ (b) xe^{-x^2} (c) e^{x^2} (d) e^{-x^2}

Soln. For the existance of fourier transform of a function $f(x)$, $f(x)$ should either finite or zero at $x=\pm\infty$. But $f(x)=e^{x^2}$ is infinite at $x=\pm\infty$

Correct option is (c)

2. The Fourier transform the function $f(x)$ is $F(k)=\int e^{ikx}f(x)\,dx$. The Fourier transform of $\frac{df(x)}{dx}$ is [GATE 2003]

(a) $\frac{dF(x)}{dk}$ (b) $\int F(k)/dk$ (c) $-ikF(k)$ (d) $ikF(k)$

Soln. $F\left[\frac{df}{dx}\right]=\int_{-\infty}^{\infty}\frac{df}{dx}e^{ikx}dx$

$$=e^{ikx}\cdot f(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}ike^{ikx}\cdot f(x)\,dx=0-ik\int_{-\infty}^{\infty}F(x)e^{ikx}dx=-ikF(k)$$

Correct option is (c)

3. The fourier transform $F(k)$ of a function $f(x)$ is defined as

$$F(k)=\int_{-\infty}^{\infty} dx f(x)\exp(ikx)$$

Then $F(k)$ for $f(x)=\exp(-x^2)$ is **[GATE 2004]**

[Given: $\int_{-\infty}^{\infty}\exp(-x^2)dx=\sqrt{\pi}$]

Soln. $F\left[e^{-x^2}\right]=\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{ikx}dx=\int_{-\infty}^{\infty} e^{-x^2+ikx}dx=\sqrt{\pi}\exp\left[-\frac{k^2}{4}\right]$

Correct option is (b)

4. The kth fourier component of $f(x)=\delta(x)$ is **[GATE 2006]**

(a) 1 (b) zero (c) $(2\pi)^{-1/2}$ (d) $(2\pi)^{-3/2}$

Soln. k^{th} Fourier component of $\delta(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}$

Correct option is (c)

5. If the Fourier transform $F[\delta(x-a)]=\exp(-i2\pi va)$, then $F^{-1}(\cos 2\pi va)$ will correspond to **[GATE 2008]**

(a) $\delta(x-a)-\delta(x+a)$ (b) a constant

(c) $\frac{1}{2}[\delta(x-a)+i\delta(x+a)]$ (d) $\frac{1}{2}[\delta(x-a)+\delta(x+a)]$

Soln. $F[\delta(x-a)]=\exp[-i2\pi va]$

$\Rightarrow F^{-1}[\exp(-i2\pi va)]=\delta(x-a)$

$\Rightarrow F^{-1}[\cos(2\pi va)]=\frac{1}{2}F^{-1}[\exp(i2\pi va)+\exp(-i2\pi va)]$

$$=\frac{1}{2}[\delta(x+a)+\delta(x-a)]$$

Correct option is (d)

6. The Heaviside function is defined as $H(t)=\begin{cases}+1 & \text{for } t>0\\ -1 & \text{for } t<0\end{cases}$ and its Fourier transform is given by $-2i/\omega$. The Fourier transform of $\frac{1}{2}\left[H\left(t+\frac{1}{2}\right)-H\left(t-\frac{1}{2}\right)\right]$ is **[GATE 2015]**

(a) $\frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2}$ (b) $\frac{\cos\left(\frac{\omega}{2}\right)}{\omega/2}$ (c) $\sin\left(\frac{\omega}{2}\right)$ (d) 0

Soln. Shifting property:

If $F[f(t)]=f(\omega)$, then $F[f(t\pm a)]=e^{\pm i\omega a}f(\omega)$

$F[H(t)]=-\frac{2i}{\omega}$

$F\left\{\frac{1}{2}H\left(t+\frac{1}{2}\right)-\frac{1}{2}H\left(t-\frac{1}{2}\right)\right\}$

$=\frac{1}{2}e^{-i\omega/2}\left(-\frac{2i}{\omega}\right)-\frac{1}{2}e^{i\omega/2}\left(-\frac{2i}{\omega}\right)$

$=\frac{i}{\omega}\left[e^{i\omega/2}-e^{-i\omega/2}\right]=\frac{i}{\omega}\left(2i\sin\frac{\omega}{2}\right)=-\frac{\sin\frac{\omega}{2}}{(\omega/2)}$

Correct option is (a)

PRACTICE SET

1. Find the fourier transform of:

(i) $f(x)=1-x^2$ for $|x|\le 1$

$=0$ for $|x|>1$

(ii) $f(x)=e^{-x^2/2}$ for $-\infty<x<\infty$

(iii) $f(x)=e^{-a|x|}$ for $-\infty<x<\infty\ (a>0)$

(iv) $f(x)=1+\frac{x}{a}$ for $(-a<x<0)$

$=1-\frac{x}{a}$ for $(0<x<a)$

$=0$ otherwise

2. Find the fourier sine transform of:

(i) $f(x)=\frac{1}{x}$ (ii) $f(x)=\frac{e^{-mx}}{x}$

3. Find the fourier cosine transform of:

(i) $f(x)=e^{-a^2x^2}$ (ii) $f(x)=\cos x$ $(0<x<a)$

$=0$ $(x>a)$

Chapter 7

LAPLACE TRANSFORM

7.1 Basic Formula

Laplace transform of function $f(x)$ defined in the interval $(0<x<\infty)$:

$$f(s)=L[f(x)]=\int_0^\infty e^{-sx}f(x)\,dx$$

Conditions for the existance of Laplace transform:

(a) $f(x)$ should be piecewise continuous in every finite interval.

(b) $f(x)$ is of exponential order s as $x\to\infty$ i.e $\underset{x\to\infty}{Lt}\ e^{-sx}f(x)=$ finite quantity.

Important Laplace Transforms:

(a) $L(1)=\frac{1}{s}$

(b) $L(x^n)=\frac{n!}{s^{n+1}}\ (n=0,1,2,.........)$

(c) $L(e^{ax})=\frac{1}{s-a}\ (s>a)$

(d) $L(\cos ax)=\frac{s}{s^2+a^2}\quad (s>0)$

(e) $L(\sin ax)=\frac{a}{s^2+a^2}\quad (s>0)$

(f) $L(\cosh ax)=\frac{s}{s^2-a^2}\quad (s^2>a^2)$

(g) $L(\sinh ax)=\frac{a}{s^2-a^2}\quad (s^2>a^2)$

7.2 Properties of Laplace Transform:

(1) Linear Property:

$$L[a_1f_1(x)+a_2f_2(x)]=a_1L[f_1(x)]+a_2L[f_2(x)]$$

(2) Shifting Property:

If $L[f(x)]=f(s)$, then $L[e^{ax}f(x)]=f(s-a)$

Example: $L[\cos ax]=\frac{s}{s^2+a^2}$, then $L\left[e^{bx}\cos ax\right]=\frac{(s-b)}{(s-b)^2+a^2}$

(3) Scaling Property:

If $L[f(x)]=f(s)$, then $L[f(ax)]=\frac{1}{a}f\left(\frac{s}{a}\right)$

Example: $L\left[\cos^2 x\right]=\frac{s^2+2}{s\left(s^2+4\right)}$,

then $L\left[\cos^2(ax)\right]=\frac{1}{a}\frac{(s/a)^2+2}{(s/a)\left[(s/a)^2+4\right]}=\frac{s^2+2a^2}{s\left(s^2+4a^2\right)}$

(4) If $L[f(x)]=f(s)$, then $L\left[x^n f(x)\right]=(-1)^n\frac{d^n}{ds^n}[f(s)]$

Example: $L[\sin 3x]=\frac{3}{s^2+9}$,

then $L[x\sin 3x]=(-1)^1\frac{d}{ds}\left[\frac{3}{s^2+9}\right]=(-1)\left[-\frac{3(2s)}{\left(s^2+9\right)^2}\right]=\frac{6s}{\left(s^2+9\right)^2}$

(5) If $L[f(x)]=f(s)$, then $L\left[\frac{f(x)}{x}\right]=\int_s^\infty f(s)\,ds$

Example: $L[\sin x]=\frac{1}{s^2+1}$.

then $L\left[\frac{\sin x}{x}\right]=\int\limits_s^\infty\frac{1}{s^2+1}ds=\left[\tan^{-1}s\right]_s^\infty=\frac{\pi}{2}-\tan^{-1}s=\cot^{-1}s$

(6) If $L[f(x)]=f(s)$, then $L\left[\int\limits_0^t f(x)\,dx\right]=\frac{1}{s}f(s)$

Example: $L[\sin t]=\frac{1}{s^2+1}$.

then $L\left[\int\limits_0^s\frac{\sin x}{x}dx\right]=\frac{1}{s}L\left[\frac{\sin x}{x}\right]=\frac{1}{s}\cot^{-1}s$

(7) Laplace transforms of derivatives of a function:

If $L[f(x)]=f(s)$, then

(a) $L[f'(x)]=sL[f(x)]-f(x=0)$

(b) $L[f''(x)]=s^2L[f(x)]-f'(x=0)-sf(x=0)$

(c) $L\left[f^{(n)}(x)\right]=s^n f(s)-s^{n-1}f(x=0)-s^{n-2}f'(x=0)\ldots\ldots sf^{(n-2)}(x=0)-f^{(n-1)}(x=0)$

(9) Laplace transform of periodic functions:

Suppose $f(x)$ is a periodic function having period 'T' such that

$$f(x+nT)=f(x)\quad\text{for } n=1,2,3\ldots..$$

then $L[f(x)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-sx}f(x)\,dx$

Example: Given $f(x)=x^2$ for $0<x<2$ and $f(x+2)=f(x)$

Then, $L[f(x)]=\frac{1}{1-e^{-2s}}\int\limits_0^2 e^{-sx}x^2\,dx$

$$=\frac{1}{1-e^{-2s}}\left[-e^{-sx}\frac{x^2}{s}+2e^{-sx}\left(-\frac{x}{s^2}-\frac{1}{s^3}\right)\right]_0^2$$

$$=\frac{1}{1-e^{-2s}}\frac{2}{s^3}\left[1-e^{-2s}\left(1+2s+2s^2\right)\right]$$

(10) Laplace transform of unit step function:

$$u(x-a)=u_a(x)=\begin{cases}0 & \text{for } x<a\\ 1 & \text{for } x\geq a\end{cases}$$

then $L[u(x-a)]=\int\limits_a^\infty e^{-sx}.1.dx=\frac{e^{-sa}}{s}$

(11) Laplace transform of Dirac delta function:

$$L[\delta(x-a)]=\int\limits_0^\infty e^{-sx}\delta(x-a)\,dx=e^{-sa}$$

Example-1: Find the laplace transform of $f(x)=(1+\cos 2x)$

Soln. $L[f(x)]=\int_0^\infty e^{-sx}(1+\cos 2x)\,dx=\left(\frac{e^{-sx}}{-s}\right)_0^\infty+\int_0^\infty e^{-sx}\cos 2x\,dx$

$$=\frac{1}{s}+\frac{s}{4+s^2}=\frac{2s^2+4}{s\left(s^2+4\right)}$$

Example 2: Find the laplace transform of $f(x)=2\sin 2x\cos 4x$

Soln. $L[f(x)]=L[2\sin 2x.\cos 4x]=L[\sin 6x-\sin 2x]=\left(\frac{6}{s^2+36}-\frac{2}{s^2+4}\right)$

Example 3: Find the laplace transform of $f(x)=e^{-x}(3\sinh 2x-5\cosh 2x)$

Soln. $L[f(x)]=L(3e^{-x}\sinh 2x)-L(5e^{-x}\cosh 2x)$

$$=3\frac{2}{(s+1)^2-4}-5.\frac{(s+1)}{(s+1)^2-4}=\frac{(1-5s)}{(s^2+2s-3)} \quad \text{(Using 2nd property)}$$

Example 4: Find the laplace transform of $f(x)=\left(1+xe^{-x}\right)^3$

Soln. $f(x)=\left(1+xe^{-x}\right)^3=1+x^3e^{-3x}+3xe^{-x}+3x^2e^{-2x}$

$$L[f(x)]=L(1)+L\left(x^3e^{-3x}\right)+L\left(3x\,e^{-x}\right)+L\left(3x^2e^{-2x}\right)$$

$$=\frac{1}{s}+(-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)+3(-1)^1\frac{d}{ds}\left(\frac{1}{s+1}\right)+3(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) \quad \text{(Using 4th property)}$$

$$=\frac{1}{s}+\frac{6}{(s+3)^4}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}$$

Example 5: Find the laplace transform of $f(x)=\frac{1-e^x}{x}$

Soln. $L\left(\frac{1-e^x}{x}\right)=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s-1}\right)ds=\left[\ell n\, s-\ell n(s-1)\right]_s^\infty=\left[\ell n\frac{s}{s-1}\right]_s^\infty$ (Using 5th property)

$$=\left[\ell n\, s-\ell n(s-1)\right]_s^\infty=\left[\ell n\frac{s}{s-1}\right]_s^\infty=\ell n\left(\frac{1}{1-\frac{1}{s}}\right)_s^\infty$$

$$=0-\ell n\frac{s}{s-1}=\ell n\frac{(s-1)}{s}$$

Example 6: If Laplace transform of the function $f(x)=\frac{\sin x}{x}$ is $\tan^{-1}\frac{1}{s}$, the find the Laplace transform of the function $\frac{\sin 2x}{x}$.

Soln. Since, $L\left[\frac{\sin x}{x}\right]=\tan^{-1}\frac{1}{s}=f(s)$, then $L\left[\frac{\sin 2x}{2x}\right]=\frac{1}{2}f\left(\frac{s}{2}\right)=\frac{1}{2}\tan^{-1}\frac{2}{s}$ (Using 3rd property)

So, $L\left[\frac{\sin 2x}{x}\right]=\tan^{-1}\frac{2}{s}$

Example 7: Find the Laplace transform of the Bessel function of zeroth order $J_0(x)$.

Soln. $J_0(x)=\left[1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\dots\dots\dots\right]$

$$\Rightarrow L[J_0(x)]=\left[\frac{1}{s}-\frac{1}{2^2}\frac{2!}{s^3}+\frac{1}{2^2.4^2}\frac{4!}{s^5}-\frac{1}{2^2.4^2.6^2}\frac{6!}{s^7}+\dots\dots\dots\right]$$

$$=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1.3}{2.4}\left(\frac{1}{s^2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{s^2}\right)^3+\dots\dots\dots\right]=\frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}=\frac{1}{\sqrt{s^2+1}}$$

Example 8: A periodic square wave function $f(x)$ in terms of unit step function can be written as,

$$f(x)=k\left[u_0(x)-2u_a(x)+2u_{2a}(x)-2u_{3a}(x)+\dots\dots\dots\right]$$

Calculate the Laplace transform of $f(x)$.

Soln. $f(x)=k\left[u_0(x)-2u_a(x)+2u_{2a}(x)-2u_{3a}(x)+\dots\dots\dots\right]$

$$\Rightarrow L[f(x)]=k\left[\frac{1}{s}-2\frac{e^{-as}}{s}+2\frac{e^{-2as}}{s}-2\frac{e^{-3as}}{s}+\dots\dots\dots\right]$$

$$=\frac{k}{s}\left[1-2\left(e^{-as}-e^{-2as}+e^{-3as}-\dots\dots\dots\right)\right]=\frac{k}{s}\left[1-2\frac{e^{-as}}{1+e^{-as}}\right]$$

$$=\frac{k}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]=\frac{k}{s}\left[\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}\right]=\frac{k}{s}\tanh\frac{as}{2}$$

Example 9: Find the inverse laplace transform of

$$f(s)=\frac{s-2}{(s-2)^2+25}+\frac{s+4}{(s+4)^2+81}+\frac{1}{(s+2)^2+9}$$

Soln. $L^{-1}[f(s)]=L^{-1}\left[\frac{s-2}{(s-2)^2+25}\right]+L^{-1}\left[\frac{s+4}{(s+4)^2+81}\right]+L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$

$$=e^{2x}\cos 5x+e^{-4x}\cos 9x+\frac{e^{-2x}}{3}\sin 3x$$

Example 10: Find the inverse Laplace transform of $f(s)=\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)$

Soln. $f(s)=\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)=\frac{s+1}{s^2}-\frac{s+1}{s^2+1}=\frac{1}{s}+\frac{1}{s^2}-\frac{s}{s^2+1}-\frac{1}{s^2+1}$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)\right]=1+x-\cos x-\sin x$$

Example 11: Using Laplace transform, prove that

(i) $\int_0^\infty \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$ (ii) $\int_0^\infty t e^{-3t} \sin t dt = \frac{3}{50}$

Soln. (i) $L(\sin t) = \frac{1}{s^2+1^2}$ as $L(\sin at) = \frac{a}{s^2+a^2}$

This $\Rightarrow L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1} ds = \left(\tan^{-1} s\right)_s^\infty = \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right)$

$$\Rightarrow \int_0^\infty e^{-st}\left(\frac{\sin t}{t}\right) dt = \tan^{-1}\left(\frac{1}{s}\right)$$

Putting s = 0, we get $\int_0^\infty \left(\frac{\sin t}{t}\right) dt = \tan^{-1}(\infty) = \frac{\pi}{2}$

(ii) Since, $L(\sin t) = \frac{1}{s^2+1}$ and so $L(t\sin t) = (-1)^1 \frac{d}{ds} L(\sin t)$

$$\Rightarrow \quad L(t\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{\left(s^2+1\right)^2}$$

$$\Rightarrow \quad \int_0^\infty e^{-st}(t\sin t) dt = \frac{2s}{\left(s^2+1\right)^2}$$

Putting s = 3, $\int_0^\infty e^{-3t}(t\sin t dt) = \frac{2\times 3}{\left(3^2+1\right)^2} = \frac{3}{50}$

Example 12: Find the Laplace transform of (i) $\sin\sqrt{t}$, (ii) $\cos\sqrt{t}/\sqrt{t}$

Soln. (i) we know that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\ldots$$

$$\therefore \quad \sin\sqrt{t} = \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \ldots..$$

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, \qquad \Gamma(n+1) = n\Gamma(n), \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$L\left(\sin\sqrt{t}\right) = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{s^{5/2}(3)!} + \frac{\Gamma(7/2)}{s^{7/2}(5)!} - \frac{\Gamma(9/2)}{s^{9/2}(7)!} + \ldots\ldots$$

$$= \frac{\sqrt{\pi}}{s^{3/2}}\left[\frac{1}{2} - \frac{1}{3!}.\frac{3}{2}.\frac{1}{2}.\frac{1}{s} + \frac{1}{s^2(5)}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2} - \frac{1}{s^3.7!}\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2} + \ldots..\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2}\left[1 - \frac{1}{2^2 s} + \frac{1}{\left(2^2 s\right)^2 (2)!} - \frac{1}{\left(2^2 s\right)^3 (3)!}\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2}\left[e^{-1/2^2 s}\right] = \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s} = f(s)(say)$$

(ii) Let $L\{F(t)\} = L\left\{\sin\sqrt{t}\right\} = \frac{\sqrt{\pi}\, e^{-1/4s}}{2s^{3/2}} = f(s)$

$$F(0) = \sin 0 = 0$$

$$F'(t) = \frac{d}{dt}\left(\sin\sqrt{t}\right) = \left(\frac{1}{2\sqrt{t}}\cos\sqrt{t}\right) = \frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$\Rightarrow \quad L\{F'(t)\} = sf(s) - F(0) = s f(s) - 0 = s f(s) = \frac{\sqrt{\pi}}{2s^{1/2}} e^{-1/4s}$$

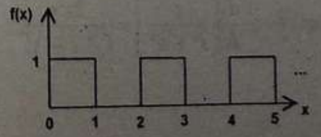
$$\Rightarrow \quad \frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{1}{2}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s} \qquad \Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

CSIR-NET/JRF Previous Years Question with Solutions

1. The graph of the function

$$f(x) = \begin{cases} 1 & \text{for } 2n \le x \le 2n+1 \\ 0 & \text{for } 2n+1 \le x \le 2n+2 \end{cases}$$

(Where n = 0, 1, 2, ...) is shown below.



Its Laplace transform $\tilde{f}(s)$ is

(a) $\frac{1+e^{-s}}{s}$ (b) $\frac{1-e^{-s}}{s}$ (c) $\frac{1}{s\left(1+e^{-s}\right)}$ (d) $\frac{1}{s\left(1-e^{-s}\right)}$

[CSIR DEC 2011]

Soln. Given function is a periodic function of period T = 2.

$$L[f(x)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-sx} f(x) dx = \frac{1}{1-e^{-2s}}\int_0^2 e^{-sx} f(x) dx$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-sx} dx\right] = \frac{1}{1-e^{-2s}}\left(\frac{1-e^{-s}}{s}\right)$$

$$=\frac{\left(1-e^{-s}\right)}{s\left(1+e^{-s}\right)\left(1-e^{-s}\right)}=\frac{1}{s\left(1+e^{-s}\right)}$$

Correct answer is (c).

2. The inverse Laplace transform of $\frac{1}{s^2(s+1)}$ is

(a) $\frac{1}{2}t^2e^{-t}$ (b) $\frac{1}{2}t^2+1-e^{-t}$ (c) $t-1+e^{-t}$ (d) $\frac{1}{2}t^2(1-e^{-t})$

[CSIR JUNE 2013]

Soln. $f(s)=\frac{1}{s^2(s+1)}=\frac{(s+1)-s(s+1)+s^2}{s^2(s+1)}=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}$

$\Rightarrow L^{-1}\left[f(s)\right]=t-1+e^{-t}$

Correct answer is (c).

GATE Previous Years Question with Solutions

1. The laplace transform of $f(t)=\sin\pi t$ is $F(s)=\frac{\pi}{\left(s^2+\pi^2\right)}, s>0$. Therefore, the laplace transform of $t\sin\pi t$ is: **[GATE 2004]**

(a) $\frac{\pi}{s^2\left(s^2+\pi^2\right)}$ (b) $\frac{2\pi}{s^2\left(s^2+\pi^2\right)}$ (c) $\frac{2\pi s}{\left(s^2+\pi^2\right)^2}$ (d) $\frac{2\pi}{\left(s^2+\pi^2\right)^2}$

Soln. **Property :** $L\left[f(x)\right]=f(s)$ then $L\left[x^n f(x)\right]=(-1)^n\frac{d^n}{ds^n}\left[f(s)\right]$

$$L[t\sin\pi t]=(-1)^1\frac{d}{ds}\left[L(\sin\pi t)\right]=-\frac{d}{ds}\left[\frac{\pi}{s^2+\pi^2}\right]=\frac{2\pi s}{\left(s^2+\pi^2\right)^2}$$

Correct option is (c)

2. If $\vec{f}(s)$ is the Laplace transform of $f(t)$ the Laplace transform of $f(at)$, where a is a constant is,

(a) $\frac{1}{a}\vec{f}(s)$ (b) $\frac{1}{a}\vec{f}(s/a)$ (c) $\vec{f}(s)$ (d) $\vec{f}(s/a)$ **[GATE 2005]**

Soln. $L\left[f(t)\right]=\vec{f}(s)=\int_0^\infty e^{-st}f(t)\,dt$

$L\left[f(at)\right]=\int_0^\infty e^{-st}f(at)\,dt=\int_0^\infty e^{-s\cdot\frac{z}{a}}f(z)\frac{dz}{a}$ [Putting, $at=z\Rightarrow a\cdot dt=dz$] $=\frac{1}{a}\vec{f}(s/a)$

Correct option is (b)

3. Inverse Laplace transformation of $\frac{s+1}{s^2-4}$ is **[GATE 2007]**

(a) $\cos 2x+\frac{1}{2}\sin 2x$ (b) $\cos x+\frac{1}{2}\sin x$

(c) $\cosh x+\frac{1}{2}\sinh x$ (d) $\cosh 2x+\frac{1}{2}\sinh 2x$

Soln. $f(s)=\frac{s+1}{s^2-4}=\frac{s}{s^2-4}+\frac{1}{s^2-4}$

$L^{-1}\left[f(s)\right]=L^{-1}\left[\frac{s}{s^2-1}\right]+L^{-1}\left[\frac{1}{s^2-4}\right]$

$=\cosh 2x+\frac{1}{2}\sinh 2x$

Correct option is (d)

4. If $f(x)=\begin{cases}0 & for\ x<3\\ x-3 & for\ x\geq 3\end{cases}$ then, the laplace transform of $f(x)$ is **[GATE 2010]**

(a) $s^{-2}e^{3s}$ (b) s^2e^{-3s} (c) s^{-2} (d) $s^{-2}e^{-3s}$

Soln. $L\left[f(x)\right]=\int_3^\infty e^{-sx}(x-3)\,dx=(x-3)\frac{e^{-sx}}{-s}\Big|_3^\infty-\int_3^\infty 1\cdot\frac{e^{-sx}}{-s}dx=-\frac{e^{-sx}}{s^2}\Big|_3^\infty=\frac{e^{-3s}}{s^2}$

Correct option is (d)

5. Which of the following pairs of the given function $F(t)$ and its Laplace transform $f(s)$ is not correct? **[GATE-2013]**

(a) $F(t)=\delta(t), f(s)=1$ (singularity at 0) (b) $F(t)=1, f(s)=\frac{1}{s}(s>0)$

(c) $F(t)=\sin kt, f(s)=\frac{s}{s^2+k^2}(s>0)$ (d) $F(t)=te^{kt}, f(s)=\frac{1}{(s-k)^2}(s>k, s>0)$

Soln. $L[\delta(t)]=\int_0^\infty\delta(t)e^{-sx}dx=1$

$L[1]=\int_0^\infty 1\cdot e^{-sx}dx=\left[\frac{e^{-sx}}{-s}\right]_0^\infty=\frac{1}{s}$

$L[\sin kt]=\int_0^\infty \sin kt\, e^{-st}dt=\frac{k}{s^2+k^2}$

$L[te^{kt}]=L[t]_{s\to s-k}=\frac{1}{s^2}\Big|_{s\to s-k}=\frac{1}{(s-k)^2}$

Correct option is (c)

PRACTICE SET

1. Evaluate the Laplace transform of the following functions:

(i) $f(x) = (x-1) \quad if\ 1<t<2$
$\quad = (3-x) \quad if\ 2<t<3$

(ii) $f(x) = 3e^{3x} + 5x^4 - 4\cos(3x)$

(iii) $f(x) = \cos^2 ax$

(iv) $f(x) = \sinh(ax)\cos(ax)$

(v) $f(x) = e^{-ax}\dfrac{x^{n-1}}{(n-1)!}$

(vi) $f(x) = (x^2-3x+2)\sin 3x$

(vii) $f(x) = \sin \omega x \quad \text{if } 0<x<\dfrac{\pi}{\omega}$
$\quad = 0 \quad if\ \dfrac{\pi}{\omega}<x<\dfrac{2\pi}{\omega}$

(viii) $f(x) = \dfrac{1}{\epsilon} \quad if\ 0 \le x \le \epsilon$
$\quad = 0 \quad if\ x > \epsilon$

(ix) $f(x) = \dfrac{\sin 2x}{x}$

$$\left[\text{Ans. } (i)\ \frac{1}{s^2}\left(e^{-s}-2e^{-2s}+e^{-3s}\right), (ii)\ \frac{3}{s-3}+\frac{120}{s^5}-\frac{4s}{s^2+9}, (iii)\ \frac{s^2+2a^2}{s\left(s^2+4a^2\right)}, (iv)\ \frac{a\left(s^2-2a^2\right)}{s^4+4a^4},\right.$$

$$(v)\ \frac{1}{(s+a)^n}, (vi)\ -\frac{18s^2+54}{\left(s^2+9\right)^3}-\frac{18s}{\left(s^2+9\right)^2}+\frac{6}{\left(s^2+9\right)} (vii)\ \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left(\frac{\omega e^{-\frac{s\pi}{\omega}}+\omega}{s^2+\omega^2}\right),$$

$$\left.(viii)\ \frac{1}{\epsilon s}\left(1-e^{-\epsilon s}\right), (ix)\ \cot^{-1}\frac{s}{2}\right]$$

2. Laplace transform of $f'(x)$ can be represented as

(a) $sL[f(x)]$ (b) $sL[f(x)] - f(0)$ (c) $L[f(x)] - f(0)$ (d) $L[f(x)] - f(1)$

3. Laplace transform of $f''(x)$ can be represented as

(a) $L^2[f(x)]$

(b) $s^2L^2[f(x)] - f(0)$

(c) $s^2L[f(x)] - f'(0) - sf(0)$

(d) $s^2L[f(x)]$

4. The value of the integral $\int_0^\infty xe^{-2x}\sin x\,dx$ is equal to

(a) $\dfrac{4}{25}$ (b) $\dfrac{4}{50}$ (c) $\dfrac{2}{15}$ (d) $\dfrac{3}{35}$

5. If the error function is defined as: $erf(t) = \dfrac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$, then the laplace transform of $erf\left(\sqrt{t}\right)$ is equal to

(a) $\dfrac{1}{\sqrt{s(s+1)}}$ (b) $\dfrac{1}{\sqrt{(s+1)}}$ (c) $\dfrac{1}{s\sqrt{s(s+1)}}$ (d) $\dfrac{1}{s\sqrt{(s+1)}}$

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6. The inverse laplace transform of $f(s) = \dfrac{s-2}{(s-2)^2+25} + \dfrac{s+4}{(s+4)^2+81} + \dfrac{1}{(s+2)^2+9}$ is

(a) $e^{-2x}\cos 5x + e^{-4x}\cos 9x + \dfrac{e^{-2x}}{3}\sin 3x$

(b) $e^{2x}\cos 5x + e^{-4x}\cos 9x + \dfrac{e^{-2x}}{3}\sin 3x$

(c) $e^{2x}\cos 5x + e^{4x}\cos 9x + \dfrac{e^{2x}}{3}\sin 3x$

(d) $e^{2x}\cos 5x + e^{4x}\cos 9x + \dfrac{e^{-2x}}{3}\sin 3x$

7. The inverse laplace transform of $f(s) = \dfrac{1}{(s-4)^5} + \dfrac{5}{(s-2)^2+25}$ is

(a) $e^{4x}\dfrac{x^4}{12} - e^{2x}\cos 5x$

(b) $e^{4x}\dfrac{x^4}{12} + e^{2x}\cos 5x$

(c) $e^{4x}\dfrac{x^4}{24} + e^{2x}\sin 5x$

(d) $e^{4x}\dfrac{x^4}{12} - e^{2x}\sin 5x$

8. The inverse Laplace transform of $f(s) = \dfrac{1}{s^2(s+1)^2}$ is

(a) $te^{-t} + 2e^{-t} + t + 2$

(b) $te^{-t} + 2e^{-t} + t - 2$

(c) $te^{-t} + e^{-t} + t - 2$

(d) $te^{-t} - e^{-t} + t - 2$

9. The inverse Laplace transform of $f(s) = \dfrac{3s+7}{s^2-2s-3}$ is

(a) $2e^{3x} - e^{-x}$ (b) $4e^{3x} + e^{-x}$ (c) $2e^{3x} + e^{-x}$ (d) $4e^{3x} - e^{-x}$

Chapter 8

PROBABILITY

8.1 Basic Review of Probability

Definition:

Probability is the numerical measurement of degree of certainity of the occurrence of an event. If an event A can happen in m ways and fail in n ways then the probability of happening of event A will be equal to

$$P(A)=\frac{m}{m+n}$$

and probability of failing of event $A = P'(A)=\frac{n}{m+n}$

Therefore, $P(A)+P'(A)=1$

$$\text{Probability of any event } A=\frac{\text{number of events or ways favourable to A}}{\text{total number of events or ways}}$$

(i) Sample Space: The set at all possible outcomes in a single performance of an random experiment is known as sample space. Each outcome is called sample point.

Example: Tossing a coin, we can get head or tail. Therefore, sample space S = (H, T)

(ii) Random experiment : These are experiments, in which result may be altogether different, even though they are performed under identical conditions.

Example: Tossing a coin, Throwing a die

(iii) Trial and Event: Performing a random experiment is called 'trial' and outcome is called 'event'.

Example: Tossing a coin is a trial and turning up head /tail are events.

(iv) Equally like events: Two events are said to be equally like events, if one of them cannot be expected in preference to the other.

Example: Drawing a card from a well shuffled pack,we may get any card, then all 52 different cases are equally like.

(v) Independent Events: If happening of one event does not influence occurrence of other, then they are independent events.

Example: If two coins are simultanously, then the event of getting head on first coin and the event of getting tail on second coin are independent.

(vi) Mutually Exclusive Events: Two events are known as mutually exclusive events, when the occurrence of one of them excludes the occurance of the other.

Example: Tossing a coin, we can get either head or tail but not both. Therefore, getting a head and getting a tail are mutually exclusive events.

(vii) Conditional Probability: The probability of happening of event A, such that B has already occurred, is called conditional probability of happening of A. It is usually denoted as P(A | B)

Example 1: What is the probability of throwing an even number in case of a dice?

Soln: $P=\frac{3\,[\text{getting 2, 4, 6 will be favourable cases}]}{6\,[\text{total no. of outcomes possibles are 1, 2, 3, 4, 5, 6}]}$

Example 2: A bag contains 7 red balls and 8 black balls. Find the probability of drawing a red ball.

Soln: $P=\frac{{}^{7}c_1[\text{No. of ways in which 1 red ball is chosen from 7 red balls}]}{{}^{15}c_1\,[\text{No. of ways in which 1 ball is chosen from 15 balls}]}$

Example 3: A unbiased coin is tossed two times. What is the probability of at least one head?

Soln: $P=\frac{3\left[\text{Outcomes having at least one head are: HT, TH,HH}\right]}{4\left[\text{Different outcomes possible: HH,HT,TH,TT}\right]}$

Theorem 1: Probability that either event A or event B or both will occur (when they are mutually exclusive) is given by

$$P(A \cup B) = P(A) + P(B)$$

Theorem 2: Probability that either event A or event B or both will occur(when they are not mutually exclusive) is given by,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Theorem 3: Probability that both A, B events occur (when they are independent events) is given by,

$$P(A \cap B) = P(A)\,P(B)$$

Theorem 4: Probability that both A, B events occur (when they ar not independent events) is given by,

$$\begin{aligned} P(A\cap B) &= P(B|A).P(A) \quad [\text{Here A occurs first, then } B] \\ &= P(A|B).P(B) \quad [\text{Here B occurs first, then } A] \end{aligned}$$

Theorem 5: For every event A, $0 \le P(A) \le 1$ i.e. a probability is between 0 and 1.

Theorem 6: $P(\varnothing)=0$ i.e. the impossible event has probability zero.

Theorem 7: If A' is the complement of A, then $P(A')=1-P(A)$

Example 4: Find the probability of throwing 9 with two dice.

Soln: Total number of possible ways of throwing two dice $=6\times6=36$

Number of ways getting 9 i.e. (3+6), (4+5), (5+4), (6+3) = 4

Therefore, the required probability = 4/36 = 1/9

Example 5: What is the probability that a leap year, selected at random, will contain 53 sundays?

Soln: Every year has minimum of 52 weeks = 364 days i.e. there will be 52 sundays, 52 mondays, ... etc. Since a leap year contains 366 days, there are 2 extra days. If the year starts on either saturday or sunday, we will get the 53rd sunday. Therefore, probability of getting 53 sundays in a leap year is $\frac{2}{7}$.

Example 6: A unbiased coin is tossed 3 times. What is the probability of obtaining two heads?

Soln: The possible outcomes are HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

The favorable outcomes are HHT, HTH, THH.

The required probability of getting 2 heads is $\frac{3}{8}$.

Example 7: A problem is given to three students A, B and C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that this problem will be solved?

Soln: Since $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(C)=\frac{1}{4}$, then $P'(A)=\frac{1}{2}, P'(B)=\frac{2}{3}, P'(C)=\frac{3}{4}$

The problem will not be solved if all three of them will not be able to solve the problem.

Then, probability that the problem will not be solved = $P'(A)\,P'(B)\,P'(C)=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$

Therefore, the probability that the problem will be solved = $1-\frac{1}{4}=\frac{3}{4}$

Example 8: From a pack of 52 cards one are drawn at random. What is the probability that the card is either a king or a spade?

Soln: Probability that one spade will be choosen from a pack of 52 cards = $\frac{13}{52}=\frac{1}{4}$

Probability that one king will be choosen from a pack of 52 cards = $\frac{4}{52}=\frac{1}{13}$

Since, there is one card which is both king and spade. Then, the probability that the choosen card is either a king or a spade = $\frac{1}{4}+\frac{1}{13}-\frac{1}{52}=\frac{4}{13}$

Example 9: Let $P\begin{bmatrix}1 & a\\ b & c\end{bmatrix}$, where a, b, c are chosen randomly from the set $\{1, 2, 3, 4, 5\}$. What is the probability that P is singular?

Soln: $P=\begin{bmatrix}1 & a\\ b & c\end{bmatrix}$; for the matrix to be singular $|P|=c-ab=0$

Required probability = $\frac{\text{number of singular matrices}}{\text{total number of matrices formed}}$

Each a, b, c can be chosen in five ways, therefore, total number of matrices formed = 5×5×5=125

For the following combinations of a, b, c , $|P|=c-ab=0$:

c	1	2	2	3	3	4	4	4	5	5
a	1	1	2	3	1	1	4	2	1	5
b	1	2	1	1	3	4	1	2	5	1

So, (Probability) $P=\frac{10}{125}=\frac{2}{25}$

Example 10: A six faced die is a biased one. it is thrice more likely to show an odd number than to show an even number. It is thrown twice. What is the probability that the sum of the numbers in the two throws is even?

Soln: $\text{Prob.}(\text{odd})=\text{Prob.}(\text{even})$

$\text{Prob.}(\text{odd})+\text{Prob.}(\text{even})=1$

$\Rightarrow\ 3\text{Prob.}(\text{even})+\text{Prob.}(\text{even})=1$

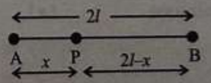
$\Rightarrow \text{Prob.}(\text{even})=\frac{1}{4}$ and $\text{Prob.}(\text{odd})=\frac{3}{4}$

Sum of the numbers will be even if both numbers will be either even or odd.

Required probability = $\text{Prob.}(\text{even-even})+\text{Prob.}(\text{odd-odd})=\left(\frac{1}{4}\times\frac{1}{4}\right)+\left(\frac{3}{4}\times\frac{3}{4}\right)=\frac{5}{8}$

Example 11: A point P is chosen at random on a line AB of length 2*l*. What is the probability of $P\left[(AP\times BP)>l^2/2\right]$?

Soln:



$$x(2\ell-x)>\ell^2/2 \quad \Rightarrow 2\ell x-x^2>\ell^2/2 \quad \Rightarrow x^2-2\ell x<-\ell^2/2$$

$$\Rightarrow x^2-2\ell x+\ell^2<\ell^2/2 \quad \Rightarrow (x-\ell)^2<\ell^2/2 \quad \Rightarrow (x-\ell)^2<\left(\frac{\ell}{\sqrt{2}}\right)^2$$

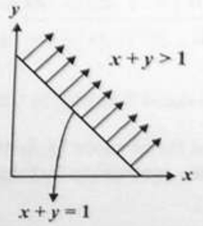
$$\Rightarrow -\frac{\ell}{\sqrt{2}}<x-\ell<\left(\frac{\ell}{\sqrt{2}}\right) \quad \Rightarrow \left(\ell-\frac{\ell}{\sqrt{2}}\right)<x<\left(\ell+\frac{\ell}{\sqrt{2}}\right)$$

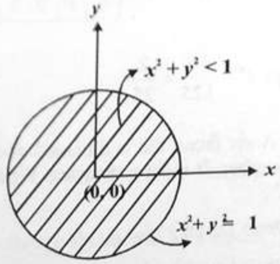
Required probability = $\frac{\text{favourable length}}{\text{total length}}=\frac{\left(\ell+\frac{\ell}{\sqrt{2}}\right)-\left(\ell-\frac{\ell}{\sqrt{2}}\right)}{2\ell}=\frac{1}{\sqrt{2}}$

Example 12: What is the probability that the sum of two numbers x and y randomly chosen in the interval (0, 1) greater than 1 while the sum of the squares less than 1?

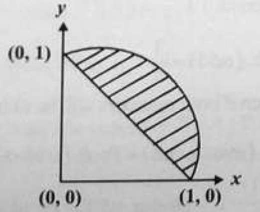
Soln: Since x and y can be choosen in the interval (0, 1), therefore, the sample space will be square of length 1 unit.

The sum of two numbers x, y will be greater than 1 i.e. $x+y>1$, if we choose the two numbers from the region above the straight line $x+y=1$ and the sum of their squares will be less than 1 i.e. $x^2+y^2<1$, if we choose the two numbers from the region within the circle $x^2+y^2=1$.





The favourable area will be as following (given by shaded region in the figure below)



$$\text{Required probability} = \frac{\text{favourable area}}{\text{sample area}} = \frac{\frac{\pi}{4}\times 1^2 - \frac{1}{2}\times 1\times 1}{1\times 1} = \frac{\pi}{4} - \frac{1}{2}$$

8.2 Baye's Theorem

If $B_1, B_2, B_3 \ldots B_n$ are mutually exclusive events (with non-zero probability) of an random experiment, then for any arbitrary event A of the sample space of above random experiment, the probability of event B_k such that A has already occured is

$$P(B_k \mid A) = \frac{P(B_k)\,P(A \mid B_k)}{\sum_{i=1}^{n} P(B_i)P(A \mid B_i)}$$

Example 13: Suppose bag 1 contains 2 white, 3 red balls and bag 2 contains 3 white and 4 red balls. If a bag is selected at random and a ball is drawn from it. If the drawn ball is white, what is probability that the ball is from bag 1?

Soln: Event A: The drawn ball is white, Event B_1: Choosing bag 1, Event B_2: Choosing bag 2

Probability that the ball is chosen from bag 1 given that the ball is white =

$$P(B_1 \mid A) = \frac{P(B_1)\,P(A \mid B_1)}{\sum_{i=1}^{2} P(B_i)P(A \mid B_i)} = \frac{P(B_1)\,P(A \mid B_1)}{P(B_1)\,P(A \mid B_1) + P(B_2)\,P(A \mid B_2)} = \frac{\frac{1}{2}\times\frac{{}^2C_1}{{}^5C_1}}{\frac{1}{2}\times\frac{{}^2C_1}{{}^5C_1} + \frac{1}{2}\times\frac{{}^3C_1}{{}^7C_1}} = \frac{14}{29}$$

Example 14: A box contains 99 fair coins and 1 double-headed coin. One coin is chosen from the box at random. The coin is tossed 3 times and we get heads all the time. What is the probability that the chosen coin will be double-headed?

Soln: Total 100 coins = 99 fair coins + 1 double headed coin

Suppose Event A : getting 3 heads in 3 tosses

Event B_1 : selecting a fair coin

Event B_2 : selecting a double headed coin.

$$P(B_2 \mid A) = \frac{P(B_2)P(A \mid B_2)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2)} = \frac{\frac{{}^1C_1}{{}^{100}C_1}\times 1}{\frac{{}^{99}C_1}{{}^{100}C_1}\times\frac{1}{8} + \frac{{}^1C_1}{{}^{100}C_1}\times 1} = 0.075$$

8.3 Random Variable

It is defined as a real number 'x' connected with the outcomes of a random experiment. There are of two types as follows:

(i) Discrete random variables

(ii) Continuous random variables

Discrete Random Variable:

In this case the random variable x will take discrete values.

Example:

Two coins are tossed together. The sample place will be

S = {HH, HT, TH, TT}

Let X be a ranodm variable which denote the number of heads that we have obtained. Then, X can take values 0, 1, 2

Continuous Random Variable:

In this case random variable x can take contiuous values.

Example : Height and weight of a students in a class is a continuous random variable

8.4 Probability Distribution

This gives the all possible outcomes of a random experiment with their respective probabilities. These are of two types

(i) Discrete probability distribution

(ii) Continuous probability distribution

Discrete probability distribution:

Let X is a discrete random variable which can take the following discrete values $X_i\,(i=1,2,3,......n)$ with corresponding probabilities $P(X_i)(i=1,2,3.....n)$ such that

$$\sum_{i=1}^{n} P(x_i)=1$$

then it will be discrete probability distribution with $P(x_i)$ as a probability distribution function.

(i) Mean of the distribution:

It can be defined as the average or expected value of the random variable 'x' and given by

$$E(x)=\mu=\langle x\rangle=\sum_{i=1}^{n} x_i P(x_i)$$

(ii) Variance and standard deviation of the distribution:

It can be defined as the measurement of deviation of the random variable from its mean value and given by

$$\text{variance } \sigma^2(x)=\sum_{i=1}^{n}(x_i-\mu)^2 P(x_i)=\langle x^2\rangle-\langle x\rangle^2$$

$$\text{Standard Deviation, } \sigma(x)=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$$

Note:

(i) $E(K\,x)=K\,E(x)$

(ii) $E(x+y)=E(x)+E(y)$

(iii) $E(xy)=E(x)\cdot E(y)$

(iv) $\sigma^2(x+y)=\sigma^2(x)+\sigma^2(y)$

Example 15: A dice thrown and the outcomes are observed. Find the mean, variance, standard deviation of the random experiment.

Soln: The probability distirbution of the random experiment is as follows

x_i	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\text{Mean } \mu=\sum_{i=1}^{6} x_i P(x_i)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=3.5$$

Variance

$$\sigma^2=\sum_{i=1}^{6}(x_i-\mu)^2 P(x_i)$$

$$=(1-3.5)^2\times\frac{1}{6}+(2-3.5)^2\times\frac{1}{6}+(3-3.5)^2\times\frac{1}{6}+(4-3.5)^2\times\frac{1}{6}+(5-3.5)^2\times\frac{1}{6}+(6-3.5)^2\times\frac{1}{6}=2.9$$

Standard deviation $\sigma=\sqrt{2.9}=1.7$

Continuous probability distribution:

Suppose x is a continuous random variable and the continuous probability distribution of x is defined by a probability density function or probability distribution function $f(x)$ such that

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

i.e. probability of finding the continuous random variable x from $-\infty$ to $+\infty$ is equal to 1.

Mean of the distribution (μ) $=\langle x\rangle=\int_{-\infty}^{\infty} x f(x)\,dx$

Variance $\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

Standard deviation $\sigma=\left[\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx\right]^{1/2}$

Example 17: Suppose, 'x' is a continuos random varible with probability density function given by,

$$f(x)=kx\,;(0\le x<2)$$
$$=2k\,;(2\le x<4)$$
$$=-kx+6k\,;(4\le x<6)$$

What is the value of k and mean of the distribution?

Soln: $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\Rightarrow\int_0^2 kx\,dx+\int_2^4 2k\,dx+\int_4^6(-kx+6k)\,dx=1 \qquad \Rightarrow k=\frac{1}{8}$$

$$\text{Mean } \mu=\int_{-\infty}^{\infty} xf(x).dx=\int_0^2 k.x^2dx+\int_2^4 2kx\,dx+\int_4^6(-kx+6x)\,x\,dx=3$$

8.5 Binomial Distribution

Binomial distribution can be used under the following conditions

(a) A random experiment is repeated many times but the number of trials will be finite

(b) There are only two possible outcomes i.e. success and failure, for each trial.

(c) All trials are independent of each other

If a random experiment is repeated 'n' times (i.e. there are 'n' trials of a random experiment), the probability that event A will occur 'r' times is

$$P(r) = {}^{n}C_r\, p^r q^{n-r}$$

where, n = total number of independent trials
r = number of success
p = probability of success in a single trial
q = probability of failure in a single trial.

Mean of distribution $= \mu = \langle x \rangle = np$

Standard Deviation of the ditribution $\sigma = \sqrt{npq}$

Variance of the distribution $\sigma^2 = npq$

Note : (i) $\langle x \rangle = np$

(ii) $\langle x^2 \rangle = np(1-p+np)$

(iii) $\langle x^3 \rangle = np\left(1-3p+3np+2p^2-3np^2+n^2p^2\right)$

Example 18: Find the proability of getting at least 4 head in 5 tosses of a fair coin.

Soln: Number of trials $n = 5$, number of success $r = 4$ or 5.

$$\text{Required Probability} = P(4)+P(5) = {}^{5}C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4} + {}^{5}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{5-5} = 5.\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 = \frac{3}{16}$$

Example 19: Assuming that each face of six faced dice is equally likely to land uppermost. Consider a game which involves the tossing of five such dices. Calculate the probability that the number 4 appears uppermost.

(i) In exactly one dice.
(ii) In exactly two dice.
(iii) In exactly five dice
(iv) in atleast one dice.

Soln: (i) $P = {}^{5}c_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 = 0.4$ (ii) $P = {}^{5}c_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 0.16$ (iii) $P = {}^{5}c_5\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0 = \left(\frac{1}{6}\right)^5$

(iv) At least one means it will 1,2,3,4,5 . Therefore, $P = 1 - {}^{5}c_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 = 0.6$

8.6 Poisson Distribution

Poisson distribution is a limiting form of binomial distribution. It can be used under the following conditions:

(a) the number of trials are infinitely large i.e $n \to \infty$

(b) the constant probability of success for each trial is infinitely small i.e. $p \to 0$

(c) np is a finite quantity

If a random experiment is repeated 'n' times, then the probability that event A will occur 'r' times

$$P(r) = \frac{e^{-m}\cdot m^r}{r!}$$

where 'm' is the mean of the distribution.

Mean of the distribution, $\mu = \langle x \rangle = np$

Variance of the ditribution $\sigma^2 = np$

Standard Deviation of the distribution $\sigma = \sqrt{np}$

Note: (i) $\langle x \rangle = np$

(ii) $\langle x^2 \rangle = np(1+np)$

(iii) $\langle x^3 \rangle = np\left(1+3np+n^2p^2\right)$

Example 20: Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly 2 will be defective, by using (a) the binomial distribution, (b) the poisson distribution.

Soln: (a) The probability of a defective tool is p = 0.1. Let X denote the number of defective tools out of 10 chosen. Then, according to the binomial distribution,

$$P(r=2) = \left({}^{10}c_2\right)(0.1)^2(0.9)^8 = 0.1937 \text{ or } 0.19$$

(b) We have $m = np = (10)(0.1) = 1$. Then, according to the Poisson distribution,

$$P(r=2) = \frac{m^r e^{-m}}{r!} = \frac{(1)^2 e^{-1}}{2!} = 0.1839 \text{ or } 0.18$$

Example 21: A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a poisson's distribution with mean 1.5. Calculate the number of days in a year on which (i) car is not used (ii) The number of days in a year on which some demand is refused.

Soln: Mean of the distribution 'm' = 1.5

(i) If the car is not used, then demand r=0

$$P(r) = \frac{e^{-m}m^r}{r!} \Rightarrow P(0) = \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} = 0.2231$$

Therefore, number of days in a year when there is no demand of car. $= 365 \times 0.2231 = 81.4315 = 81$ days

(ii) Some demand is refused if the number of demands is more than 2 i.e. $r > 2$.

$$P(r>2)=p(3)+p(4)+...=1-\left[P(0)+P(1)+P(2)\right]$$

$$=1-\left[\frac{e^{-1.5}(1.5)^0}{0!}+\frac{e^{-1.5}(1.5)^1}{1!}+\frac{e^{-1.5}(1.5)^2}{2!}\right]=0.1912625$$

Therefore, number of days in a year when some demand is refused =
$365\times 0.1912625=69.81=70$ days

Example 22: If the probability of unfavorable reaction from a certain injection is 0.001, what is the probability that out of 2000 individuals, more than 2 will suffer from unfavorable reaction?

Soln: Number of trials n = 2000

Probability of success in single trial p = 0.001

Probability that more than two persons will suffer from the unfavourable reaction =

$$P(r>2)=1-\left[P(0)+P(1)+P(2)\right]=1-\left[\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}\right]=1-\frac{5}{e^2}$$

Example 23: There are 20 buses passing a bus-stop randomly within an hour. What is the probability that there will be no buses within 5 minutes? (Given: $e^{-5/3}=0.19$)

Soln: Probability that a bus will pass in one minute = $p=\frac{20}{60}=\frac{1}{3}$

(here a single trial corresponds to checking that the bus is coming or not in a single minute and success corresponds to passing of bus)
Here in this question, number of trials, $n=5$ and number of success, $r=0$

The probability that there will be no buses in 5 minutes = $P(r=0)=\frac{e^{-m}\ m^r}{r!}=\frac{e^{-np}\ (np)^r}{r!}=0.19$

8.7 Gaussian Distribution

The Gaussian probability distribution function for a continuous random variable x is given by

$$f(x)=\frac{1}{\sqrt{2\pi\sigma}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

where, μ = mean of distribution $(-\infty<\mu<\infty)$

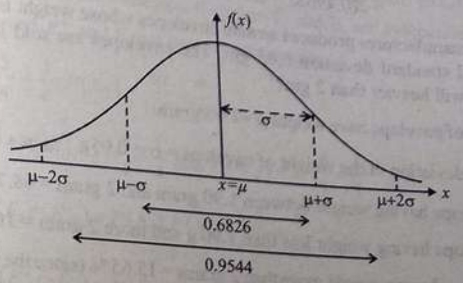
σ = S.D. of the distribution $(\sigma>0)$

Replacing $\frac{x-\mu}{\sigma}$ by z, we get the probability distribution function for the normal distribution.

i.e. $f(z)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]$

where, mean of the distribution μ = 0 and standard deviation $\sigma=1$

Normal curve and its properties:



(a) The curve is symmetric about x = μ and extends from $x=-\infty$ to $x=\infty$

(b) The curve attains a peak about $x=\mu$ and the peak value $f(x)\big|_{\max}=\frac{1}{\sqrt{2\pi\sigma}}$

(c) As we change x on both sides from μ, y decreases rapidly.

(d) Area under the curve is the measurement of probability for the ranodm variable 'x'.

$$P(\mu-\sigma\le x\le\mu+\sigma)=0.6826$$
$$P(\mu-2\sigma\le x\le\mu+2\sigma)=0.9544$$
$$P(\mu-2\sigma\le x\le\mu+3\sigma)=0.9974$$

Note: (i) $\langle x\rangle=\mu$

(ii) $\langle x^2\rangle=\mu^2+\sigma^2$

(iii) $\langle x^3\rangle=(\mu^2+3\sigma^2)\mu$

(iv) $\langle x^4\rangle=\mu^4+6\mu^2\sigma^2+3\sigma^4$

Example 24: A fair coin is tossed 500 times. Find the probability that the number of heads will not differ from 250 by (a) More than 10, (b) more than 30.

Soln: Mean and S.D. of the distribution $\mu=np=(500)\left(\frac{1}{2}\right)=250$ and $\sigma=\sqrt{npq}=\sqrt{(500)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}=11.18$

(a) We require the probability that the number of heads will lie between 240 and 260, or considering the data as continuous, between 239.5 and 260.5.

239.5 in standard untis $=\frac{239.5-250}{11.18}=-0.94$ 260.4 in standard units = 0.94

Required probability = (area under normal curve between z = −0.94 and z = 0.94)

= (twice area between z = 0 and z = 0.94) = 2 (0.3264) = 0.6528

(b) We require the probability that the number of heads will lie between 220 and 280 or, considering the data as continuous between 219.5 and 280.5

219.5 in standard units $=\frac{219.5-250}{11.18}=-2.73$ 280.5 in standard units = 2.73

Required probability = (twice area under normal curve between z = 0 and z = 2.73)
= 2(0.4968) = 0.9936

Example 25: A manufacturer produces airmail envelopes whose weight is normally distributed with a mean 1.95 gm and standard deviation 0.05 gm. The envelopes are sold in lots of 1000. How many envelopes in a lot will heavier than 2 gm?

Soln: Maximum number of envelops have weight $\mu = 1.95$ gram.

Since the standard deviation in the weight of envelops $= \sigma = 0.05\text{g} \Rightarrow \mu + \sigma = 2\text{ g}, \ \mu - \sigma = 1.90\text{ g}$

Percentage of envelops having weight between 1.90 gram and 2 gram = 68.7 %

Percentage of envelops having weight less than 1.90 g and more 2 gram = 100% − 68.7 % = 31.3%

Percentage of envelops having weight more than 2 grams = 15.65 % (since the normal curve is symmetric.)

Number of envelops having weight more than 2 grams = 1000 × 15.65 ≈ 156.

8.8 Exponential Distribution

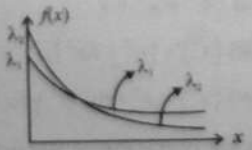
The probability distribution function is given as follows

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \ge 0$$
$$= 0 \quad \text{for } x < 0$$

where λ is known as rate parameter (+ve) and its denotes the decay rate of probability.

If λ_1, λ_2 are two rate parameter and $\lambda_2 > \lambda_1$, then

$$\text{Mean} : \mu = \langle x \rangle = \int_{-\infty}^{\infty} x\, f(x)\, dx$$



$$\langle x \rangle = \int_{-\infty}^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda} \int_{-\infty}^{\infty} t e^{-t} dt = \frac{1}{\lambda}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x} dx = \frac{1}{\lambda^2}$$

Example 26: A speaks the truth in 75% cases, and B in 80% of cases; in what percentage of cases are they likely to contradict each other in stating the same fact?

Soln: Let E_1 = the event that A speaks the truth
E_2 = the event that B speaks the truth

Then E'_1 = the event that A tells a lie
E'_2 = the event that B tells a lie

Let $C = E_1 \cap E'_2$ = the event that A speaks the truth and B tells a lie and $D = E'_1 \cap E_2$ = the event that A tells a lie and B speaks the truth.

Let E = the event that A and B contradict each other in stating the same fact, then $E = C \cup D$

Since A and B are independent witnesses, therefore E_1 and E_2 are independent events, E_1 and E'_2 are independent events and E'_1 and E_2 are independent events.

Now, $P(E_1) = \frac{75}{100} = \frac{3}{4}; P(E_2) = \frac{80}{100} = \frac{4}{5}$

$\therefore \quad P(E'_1) = 1 - P(E_1) = 1 - \frac{3}{4} = \frac{1}{4}$ and $P(E'_2) = 1 - \frac{4}{5} = \frac{1}{5}$

$$P(C) = P(E_1 \cap E'_2) = P(E_1).P(E'_2)\frac{3}{4}.\frac{1}{5} = \frac{3}{20}$$

$$P(D) = P(E'_1 \cap E_2) = P(E'_1).P(E_2) = \frac{1}{4}.\frac{4}{5} = \frac{4}{20}$$

Since C and D are mutually exclusive events, therefore,

Required probability, $P(E) = P(C \cup D) = P(C) + P(D) = \frac{3}{20} + \frac{4}{20} = \frac{7}{20}$

Example 27: Two persons A and B throw a coin alternately till one of them gets 'head' and wins the game. Find their respective probabilities of winning.

Soln: Let E = the event of occurrence of head in one throw of a coin then E' = the event of occurrence of tail in one throw of a coin.

$\therefore \quad P(E) = \frac{1}{2}$ and $P(E') = 1 - P(E) = \frac{1}{2}$

Let E_1 = the event that A wins and E_2 = the event that B wins. Clearly exactly one of E_1 and E_2 will certainly happen.

$\therefore \quad E_2 = E'_1$

For A to win, he must throw a head in first, third, fifth .... throws and B should not throw a head in second, fourth, sixth, ... throws.

$$\therefore \quad P(E_1) = \frac{1}{2} + \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right).\frac{1}{2} + \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\frac{1}{2} + \text{.... to } \infty$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 .\frac{1}{2} + \left(\frac{1}{2}\right)^4 .\frac{1}{2} + \text{.....to } \infty$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \text{..... to } \infty$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^4 + \text{.... to } \infty\right] = \frac{1}{2} \frac{1}{\left[1 - \left(\frac{1}{2}\right)^2\right]} = \left(\frac{1}{2} \times \frac{4}{3}\right) = \frac{2}{3}$$

and $P(E_2) = P(E'_1) = 1 - P(E_1) = \frac{1}{3}$. Thus, $P(E_1) = \frac{2}{3}$ and $P(E_2) = \frac{1}{3}$

Example 28: A bag contains 4 white and 6 black balls. A ball is drawn at random and replaced and then again a ball is drawn. If this is repeated 4 times, what is the probability that 3 black and 1 white ball will be drawn in these 4 trials?

Soln: When a ball is drawn out of 4 white and 6 black balls, let S be the sample space and E be the event of drawing a black ball.

Total number of balls in the bag = 10, number of black balls = 6 and number of white balls = 4

$\therefore \quad n(S) = {}^{10}C_1 = 10$ and $n(E) = {}^{6}C_1 = 6$

Here number of trials n = 4

p = probability that a black ball will be drawn in one trial

$$= \frac{n(E)}{n(S)} = \frac{6}{10} = \frac{3}{5}$$

$$\therefore \quad q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}$$

Clearly q = probability of drawing a white ball in one trial

Therefore, required probability, $P(X=3)$ or $P(3) = {}^{4}C_3 p^3 q^1 = 4\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^1 = \frac{216}{625}$

Example 29: If the chance that a ship arrives safey at a port is $\frac{9}{10}$; find the chance that out of 5 expected ships, at least 4 will arrive safety at the port?

Soln: Let E = the event that a ship will arrive safety at the port then

p = probability of occurence of event E in one trial = $\frac{9}{10}$ $\quad \therefore q = 1 - p = \frac{1}{10}$

Here number of trials = number of ships = 5

Therefore, the probability that event E will occur exactly 4 times in 5 trials is given by

$$P(4) = {}^{5}C_4\, p^4 q^1 = 5\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right)^1 = \frac{32805}{100000}$$

Probability that event E will occur exactly 5 times in 5 trials is given by

$$P(5) = {}^{5}C_5 p^5 q^0 = 1 \times \left(\frac{9}{10}\right)^5 \times 1 = \frac{59049}{100000}$$

Therefore, required probability, $= \frac{32805}{100000} + \frac{59049}{100000} = \frac{91854}{100000}$

Example 30: A bag contains n white and n red balls. Paris of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is

$$\left(\frac{2^n}{{}^{2n}C_n}\right)$$

Soln: Here random experiment is: selecting n pairs of balls from a bag containing n white and n red balls.

Let S = the sample space

and E = the event that each of the n pairs of balls drawn consists of one white and one red ball.

Now, $n(S) = \left({}^{2n}C_2\right)\left({}^{2n-2}C_2\right)\left({}^{2n-4}C_2\right)\ldots\left({}^{4}C_2\right)\left({}^{2}C_2\right)$

$$= \frac{2n!}{(2n-2)!2!} \cdot \frac{(2n-2)!}{(2n-4)!2!} \cdot \frac{(2n-4)!}{(2n-6)!2!} \cdots \frac{4!}{2!2!} \cdot 1 = \frac{(2n)!}{2^n}$$

and $n(E) = \left({}^{n}C_1 . {}^{n}C_1\right).\left({}^{n-1}C_1 . {}^{n-1}C_1\right) \ldots . \left({}^{2}C_1 . {}^{2}C_1\right).\left({}^{1}C_1 . {}^{1}C_1\right)$

$$= n^2.(n-1)^2 \ldots 2^2.1^2 = (n!)^2$$

Therefore, required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{(n!)^2 . 2^n}{(2n)!} = \left(\frac{2^n}{{}^{2n}C_n}\right)$

Example 31: A bag contains 'a' white and 'b' black balls. Two players A and B alternattively draw a ball from the bag, replacing the ball each time after the draw till one of them draws a white ball and wins the game. If A begins the game and the probability of A winning the game is three times that of B, show that $a : b = 2 : 1$

Soln: Bag contains a white b black balls.

Let E_i = the event of drawing a white ball in the i^{th} draw.

Then E'_i = the event of drawing a black ball in the i^{th} draw.

Let E = the event that A wins the game

Then (E') = the event that B wins the game

Since after each draw the ball drawn is replaced.

$$\therefore \quad P(E_i) = \frac{a}{a+b} \text{ and } P(E') = \frac{b}{a+b}$$

Now, $P(E) = P(E_1) + P(E'_1 E'_2 E_3) + P(E'_1 E'_3 E'_3 E'_4 E_5) + \ldots\ldots \text{to } \infty$

$$= P(E_1) + P(E'_1)P(E'_2).P(E_3) + P(E'_1).P(E'_2).P(E'_3).P(E'_4).P(E_5) + \ldots \text{to } \infty$$

$$= \frac{a}{a+b} + \frac{b^2}{(a+b)^2} \cdot \frac{a}{(a+b)} + \frac{b^4}{(a+b)^4} \cdot \frac{a}{a+b} + \ldots\ldots \text{to } \infty$$

$$= \left(\frac{a}{a+b}\right) \cdot \left[1 + \left(\frac{b}{a+b}\right)^2 \left(\frac{b}{a+b}\right)^4 + \ldots \text{to } \infty\right]$$

$$= \left(\frac{a}{a+b}\right) \cdot \frac{1}{\left[1 - \left(\frac{b}{a+b}\right)^2\right]} = \left[\frac{a(a+b)}{a^2 + 2ab}\right] = \frac{a+b}{a+2b}$$

$$\therefore \quad P(E') = 1 - \frac{a+b}{a+2b} = \frac{b}{(a+2b)}$$

According to question, $P(E) = 3P(E')$

$\therefore \quad \frac{a+b}{a+2b}=\frac{3b}{a+2b}$ or $a+b=3b$ or $a=2b$. Hence $a:b=2:1$

Example 32: The mean and the variance of a binomial variable X are 2 and 1, respectively. Find the probability that X takes values greater than 1

Soln: Given, mean np = 2 and variance, npq = 1 $\Rightarrow 2q=1 \Rightarrow q=1/2$

$\therefore \quad p=1-q=1-1/2=\frac{1}{2} \quad \therefore n=4$

Therefore, the binomial distribution is $\{(1/2)+(1/2)\}^4$

Now, $P(X>1)=1-\{P(X=0)+P(X=1)\}$

$$=1-(1/2)^4-{}^4C_1.(1/2^4)=1-(5/2^4)=11/16$$

CSIR-NET/JRF Previous Years Questions with Solutions

1. An unbiased dice is thrown three times successively. The probability that the number of dots on the uppermost surface add up to 16 is **[CSIR Dec 2011]**

(a) $\frac{1}{16}$ (b) $\frac{1}{36}$ (c) $\frac{1}{108}$ (d) $\frac{1}{216}$

Soln. The number of outcomes in a single throw of a unbiased dice is 6 and since it is thrown three times, then the total number of outcomes will be $6^3=216$

The combinations for which the number of dots on the upper most surface will add up to 16 are (6, 6, 4), (6, 4, 6), (4, 6, 6), (5, 5, 6), (5, 6, 5), (6, 5, 5)

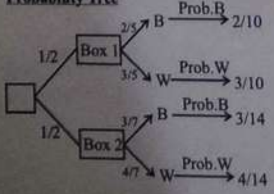
The required probability $P=\frac{6}{216}=\frac{1}{36}$

Correct answer is (b)

2. A ball is picked at random from one of the two boxes that contain 2 black and 3 white and 3 black and 4 white balls respectively. What is the probability that it is white? **[CSIR June 2012]**

(a) $\frac{34}{70}$ (b) $\frac{41}{70}$ (c) $\frac{36}{70}$ (d) $\frac{29}{70}$

Soln.



Probability Tree

(Prob B means probability of getting a black ball etc)

Probability of picking white ball $=\frac{3}{10}+\frac{4}{14}=\frac{41}{70}$

Correct answer is (b).

3. In a series of five cricket matches, one of the captains calls ‘heads’ every time when the toss is taken. The probability that he will win 3 times and lose 2 times is **[CSIR Dec 2012]**

(a) $\frac{1}{8}$ (b) $\frac{5}{8}$ (c) $\frac{3}{16}$ (d) $\frac{5}{16}$

Soln. Number of trials i.e. tosses ‘n’ = 5

Number of success i.e. getting heads ‘r’ = 3

Probability of success i.e. getting head in one trial ‘p’ = 1/2

Probability of failure i.e. getting tail in one trial ‘q’ = 1/2

Required probability: $P={}^5C_3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\frac{5!}{3!2!}\times\frac{1}{32}=\frac{5.4}{2}\times\frac{1}{32}=\frac{5}{16}$

Correct answer is (d)

4. Consider three particles A, B and C, each with an attribute S that can take two value ± 1. Let $S_A=1$, $S_B=1$ and $S_C=-1$ at a given instant. In the next instant, each S value can change to –S with probability 1/3. The probability that $S_A+S_B+S_C$ remains unchanged is **[CSIR June 2013]**

(a) 2/3 (b) 1/3 (c) 2/9 (d) 4/9

Soln. At a given instant, $S_A=1, S_B=1, S_C=-1 \Rightarrow S_A+S_B+S_C=1$

At the next instant, $S_A+S_B+S_C$ will remain unchanged for the following cases:

S_A	S_B	S_C	$S_A+S_B+S_C$
1 (unchanged)	1 (unchanged)	−1 (unchanged)	1
−1 (changed)	1 (unchanged)	1 (changed)	1
1 (unchanged)	−1 (changed)	1 (changed)	1

Required probability $=P=\left(\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\right)+\left(\frac{1}{3}\times\frac{2}{3}\times\frac{1}{3}\right)+\left(\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\right)=\frac{4}{9}$

Correct answer is (d)

5. A loaded dice has the probabilities $\frac{1}{21},\frac{2}{21},\frac{3}{21},\frac{4}{21},\frac{5}{21}$ and $\frac{6}{21}$ of turning up 1,2,3,4,5 and 6 respectively. If it is thrown twice, what is probability that the sum of the numbers that turn up is even? **[CSIR Dec 2013]**

(a) $\frac{144}{441}$ (b) $\frac{225}{441}$ (c) $\frac{221}{441}$ (d) $\frac{220}{441}$

Soln. If the loaded dice is thrown twice, the sum of the numbers that turnup will be even if both number will be even or both will be odd.

Required probability = P(Even – Even) + P(Odd – Odd) = $\left(\frac{12}{21}\times\frac{12}{21}\right)+\left(\frac{9}{21}\times\frac{9}{21}\right)=\frac{225}{441}$

Correct answer is (b)

6. A child makes a random walk on a square lattice of lattice constant a taking a step in the north, east, south, or west directions with probabilities 0.255, 0.255, 0.245, and 0.245, respectively. After a large number of steps, N, the expected position of the child with respect to the starting point is at a distance

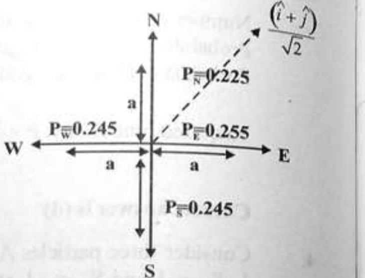
[CSIR Dec 2013]

(a) $\sqrt{2}\times10^{-2}Na$ in the north-east direction

(b) $\sqrt{2N}\times10^{-2}a$ in the north-east direction

(c) $2\sqrt{2}\times10^{-2}Na$ in the south-east direction

(d) 0



Soln. Single step length = a

After a single step, the expected value of position of the child.

$$\langle\vec{x}\rangle_1=\sum_i P_i\vec{x}_i$$

$$\Rightarrow \quad \langle\vec{x}\rangle_1=0.255\left(a\hat{i}\right)+0.255\left(a\hat{j}\right)+0.245\left(-a\hat{i}\right)+0.245\left(-a\hat{j}\right)$$

$$\Rightarrow \quad \langle\vec{x}\rangle_1=0.01a\left(\hat{i}+\hat{j}\right)$$

After 'n' steps, $\langle\vec{x}\rangle_N=0.01\,Na\left(\hat{i}+\hat{j}\right)$

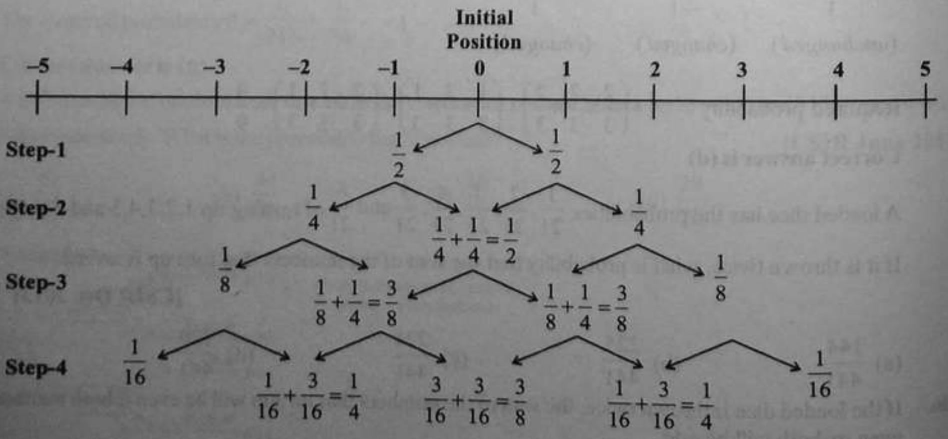
Expected position $\langle\vec{x}\rangle_N=\sqrt{2}\times10^{-2}\ Na$ (North-East Direction)

Correct answer is (a)

7. In one dimension, a random walker takes a step with equal probability to the left or right. What is the probability that the walker returns to the starting point after 4 steps?

(a) 3/8 (b) 5/16 (c) 1/4 (d) 1/16

Soln.


Initial Position
–5
–4
–3
–2
–1
0
1
2
3
4
5
Step-1
$\frac{1}{2}$
$\frac{1}{2}$
Step-2
$\frac{1}{4}$
$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$
$\frac{1}{4}$
Step-3
$\frac{1}{8}$
$\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$
$\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$
$\frac{1}{8}$
Step-4
$\frac{1}{16}$
$\frac{1}{16}+\frac{3}{16}=\frac{1}{4}$
$\frac{3}{16}+\frac{3}{16}=\frac{3}{8}$
$\frac{1}{16}+\frac{3}{16}=\frac{1}{4}$
$\frac{1}{16}$

The probability that the walker will return to the starting point after 4 steps = 3/8

Correct answer is (a)

8. Two independent random variables m and n, which can take the integer values 0, 1, 2, ...,∞, follow the Poisson distribution, with distinct mean values μ and ν respectively. Then

[CSIR Dec 2014]

(a) the probability distribution of the random variable $l=m+n$ is a binomial distribution

(b) the probability distribution of the random variable $r=m-n$ is also a Poisson distribution

(c) the variance of the random variable $l=m+n$ is equal to $\mu+\nu$

(d) the mean value of the random variable $r=m-n$ is equal to 0.

Soln. Given: m and n are two independent random variables, following Poisson distribution and the corre sponding mean values are given to be $\bar{m}=\mu$ and $\bar{n}=\nu$.

Now, consider the random variable $l=m+n$

Probability of 'L' no. of successes for the random variable l i.e

$$P(l=L)=\sum_{x=0}^{L}\frac{e^{-\bar{m}}.(\bar{m})^x}{x!}\frac{e^{-\bar{n}}.(\bar{n})^{L-x}}{(L-x)!}$$

(Here, we have assumed that out of L no. of successes of random variable z, x no. of successes correspond to random variable m and L-x no. of variables correspond to random variable n)

Therefore, $$P(l=L)=e^{-(\bar{m}+\bar{n})}\sum_{x=0}^{L}\frac{(\bar{m})^x}{x!}\frac{(\bar{n})^{L-x}}{(L-x)!}$$

Using the binomial expansion, $(\bar{m}+\bar{n})^L=\sum_{x=0}^{L}\frac{L!}{x!(L-x)!}(\bar{m})^x(\bar{n})^{L-x}$, we get

$$P(l=L)=\frac{e^{-(\bar{m}+\bar{n})}(\bar{m}+\bar{n})^L}{L!}$$

So, the random variable $l=m+n$ also follow the poisson distribution having mean value $\bar{m}+\bar{n}=\mu+\nu$ and for a poisson distribution mean and variance of the distribution is same.

9. A random walker takes a step of unit length in the positive direction with probability 2/3 and a step of unit length in the negative direction with probability 1/3. The mean displacement of the walker after n steps is

[CSIR Dec 2014]

(a) $n/3$ (b) $n/8$ (c) $2n/3$ (d) 0

Soln. If a random walker has a probability of taking a step to the right (along positive x-axis) to be p and probability of taking a step to the left (along negative x-axis) to be q, then the mean displacement of the walker after n steps will be $(p-q)n$. In the given question, $p=2/3$ and $q=1/3$. Therfore, the mean displacement of the walker after n steps = $n/3$.

Correct Option is (a)

PRACTICE SET

1. A triangle is formed by joining three points of a regular hexagon. What is the probability that the triangle will be equilateral?

(a) 1/6 (b) 1/10 (c) 1/12 (d) 1/20

2. What is the probability that the numbers formed between 60000 and 70000 using the digits 1,2,3,4,5,6,7,8,9 should end with '7'? (no digit is repeated)

(a) $\frac{2}{5}$ (b) $\frac{2}{7}$ (c) $\frac{1}{8}$ (d) $\frac{1}{11}$

3. What is probability that the words formed by the letters of the word "STRANGE" can be made so that the vowels should be in even places?

(a) $\frac{1}{7}$ (b) $\frac{2}{9}$ (c) $\frac{3}{11}$ (d) $\frac{4}{7}$

4. What is the probability that a 6-digit telephone number formed using digits 0 to 9, should start with 2 and end with 8? (no digits is repeated)

(a) $\frac{1}{32}$ (b) $\frac{1}{45}$ (c) $\frac{1}{90}$ (d) $\frac{2}{35}$

5. A bag contains 10 black and 15 red balls. Two balls are drawn from the bag at random. The probability that two balls drawn from the bag will be of same colour?

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{2}{5}$

6. Consider the experiment of throwing two fair dice. What is the probability that the sum of the numbers obtained in these dice is even ?

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

7. A bag contains 6 red and 4 green balls A fair dice is rolled and a number of balls equal to that appearing on the dice is chosen from the urn at random. The probability that all the balls selected are red is.

(a) 1/3 (b) 3/10 (c) 1/8 (d) 68/315

8. Two balls are drawn at random from a bag containing 6 white, 4 red and 5 black balls. The probability that both these balls are black, is:

(a) $\frac{1}{21}$ (b) $\frac{2}{15}$ (c) $\frac{2}{21}$ (d) $\frac{2}{35}$

9. 6 boys and 6 girls sit in a row randomly. The probability that all the girls sit together is:

(a) $\frac{1}{132}$ (b) $\frac{1}{44}$ (c) $\frac{5}{132}$ (d) $\frac{7}{132}$

10. Different words are written with the letters of PEACE. The probability that both E's come together is:

(a) $\frac{1}{3}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

11. The probability of randomly choosing 3 defectless bulbs from 15 electric bulbs of which 5 bulbs are defective is:

(a) $\frac{3}{10}$ (b) $\frac{7}{10}$ (c) $\frac{24}{91}$ (d) $\frac{67}{91}$

12. The choice of throwing 12 in a single throw with three dice is

(a) 12/216 (b) 21/216 (c) 15/216 (d) 25/216

13. The probability that a number in $\{1, 2, ..., 1001\}$ is divisible by 7 or 11 or both, is

(a) 143/1001 (b) 221/1001 (c) 247/1001 (d) 91/1001

14. A number is chosen from each of the two sets $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. If P_1 denotes the probability that the sum of the two numbers be 10 and P_2 the probability that their sum be 8, then $(P_1 + P_2)$ is

(a) 7/729 (b) 137/729 (c) 16/81 (d) 137/81

15. The probability that three men hit a target are $\frac{1}{6}, \frac{1}{4}, \frac{1}{3}$ respectively. Each shoots once the target. What is the probability that exactly one them hits the target ?

(a) $\frac{23}{96}$ (b) $\frac{42}{127}$ (c) $\frac{19}{48}$ (d) $\frac{31}{72}$

16. From a pack of 52 cards two are drawn at random. What is the probability that one is a king and the other a queen ?

(a) $\frac{5}{629}$ (b) $\frac{8}{663}$ (c) $\frac{10}{687}$ (d) None of the above

17. The probability that a number selected at random between 100 and 999 (both inclusive) will not contain the digit 3 is

(a) 16/25 (b) 9/10 (c) $(9/10)^2$ (d) 18/25

18. At a railway station a passenger leaves his luggage in a locker which is opened by dialling a three-digit code. The passenger chooses the code, closes the locker and leaves for the town. A stranger who does not know the code tries to open the locker by dialling three digits at random without repeating the unsuccessful codes. The probability that the stranger does not have to perform more than 100 trials for a success is

(a) $\frac{9}{10}$ (b) $\frac{999}{1000}$ (c) $\frac{1}{1000}$ (d) $\frac{1}{10}$

19. A cow is tied with a pole by a 100 meter long rope. What is the probability that at some point of time the cow is at least 60 meters away from the pole ?

(a) $\frac{9}{25}$ (b) $\frac{13}{25}$ (c) $\frac{16}{25}$ (d) $\frac{18}{25}$

20. A and B play a game of dice. A throws the die first, The person who first gets a 6 is the winner. What is the probability that A wins ?

(a) $\frac{6}{11}$ (b) $\frac{1}{2}$ (c) $\frac{5}{6}$ (d) $\frac{1}{6}$

21. Two teams A and B play a series of four matches. If the probability that team A wins a match is $\frac{2}{3}$, then the probability that team A wins three matches, looses one and the third win occurs in the fourth match is

(a) $\frac{8}{27}$ (b) $\frac{16}{27}$ (c) $\frac{8}{81}$ (d) $\frac{32}{81}$

22. The probabilities that a husband and wife will be alive 20 years from now are give by 0.8 and 0.9 respectively. What is the probability that in 20 years at least one, will be alive ?

(a) 0.98 (b) 0.02 (c) 0.72 (d) 0.28

23. What is the probability that an urn containing 5 balls contains only white balls if the first two balls drawn from it was found to be white

(a) 1/20 (b) 1/10 (c) 1/2 (d) 1/4

24. If the probabilities that A and B will die within a year are p and q respectively, then the probability that only one of them will be alive at the end of the year is

(a) pq (b) p(1 − q) (c) q(1 − p) (d) p + q − 2pq

25. If A and B are two events such that $P(A) = p$, $P(B) = q$, $P(A \cap B) = r$ then the probability that A occurs but B does not occur is

(a) p − r (b) p − q (c) p − qr (d) p − pq

26. A box contains n enumerated articles. All the articles are taken out one by one at random. The probability that the numbers of the selected articles are in the sequence 1, 2, ... n is

(a) $\frac{1}{n!}$ (b) $\frac{n}{n!}$ (c) $\frac{1}{n(n!)}$ (d) none of the above

27. An anti-aircraft gun can take a maximum of four shots at a plane moving away from it. The probability of hitting the plane in the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probabilities that the gun hits the plane ?

(a) 0.7325 (b) 0.7286 (c) 0.6976 (d) 0.6574

28. For n independent events $A_1, A_2, A_3, ...A_n$, let $P(A_i) = \frac{1}{i+1}$, i = 1, 2, ..., n. Then the probability that none of the events will occur is

(a) n/(n + 1) (b) n − 1/(n + 1) (c) 1/(n + 1) (d) 1/n

29. Let A and B be two events with P(A) = 1/2, P(B) = 1/3 and $P(A \cap B) = 1/4$. What is $P(A \cup B)$?

(a) 3/7 (b) 4/7 (c) 7/12 (d) 9/122

30. Let A, B and C be independent and mutually exclusive events with probability of occurrences $\frac{1-3p}{2}$, $\frac{1+4p}{3}$, $\frac{1+p}{6}$, respectively, then p lies in

(a) [− 1/4, 5/6] (b) [− 1/4, 1/3] (c) [− 1/4, 1/2] (d) [− 1/2, 1/3]

31. Three persons play a game by tossing a fair coin each independently. The game ends in a trial if all of them get the same outcome in that trial, otherwise they continue to the next trial. What is the probability that the game ends in an even number of trials ?

(a) 2/7 (b) 3/7 (c) 1/2 (d) 4/7

32. In a certain office, 1/3 of the workers are women, 1/2 of the women are married and 1/3 of the married women have children. If 3/4 of the men are married and 2/3 of the married men have children, what part of workers are without children ?

(a) 5/18 (b) 4/9 (c) 11/18 (d) 17/36

33. A and B are independent witnesses in a case. The probability that A speaks the truth is 'x' and that B speaks the truth is 'y'. If A and B agree on a certain statement, the probability that the statement is true is

(a) $\frac{xy}{xy+(1-x)(1-y)}$ (b) $\frac{xy}{(1-x)(1-y)}$

(c) $\frac{(1-x)(1-y)}{xy+(1-x)(1-y)}$ (d) $\frac{(1-x)(1-y)}{xy}$

34. A bag contains 4 white and 3 black balls and a second bag contains 3 white and 3 black balls. If a ball is drawn from each of the bags, then the probability that both are of same colour is :

(a) $\frac{3}{14}$ (b) $\frac{1}{2}$ (c) $\frac{3}{13}$ (d) $\frac{1}{3}$

35. Suppose there are two bags with first bag contaions 3 white and 2 black balls, second bag contains 2 white and 4 black balls. one ball is transferred fron bag I to bag II. and then a ball is drawn from the latter. It happens to be white. What is the probability that the tranferred ball is white?

(a) $\frac{9}{13}$ (b) $\frac{7}{13}$ (c) $\frac{6}{13}$ (d) $\frac{11}{13}$

36. Three urns are given each cotaining red and white balls. Urn 1 contains 6 red and 4 white balls, Urn II contains 2 red and 6 white balls and Urn III 1 red and 4 white balls. An urn is selected at random and a ball is drawn. If the selected ball is red, what is the probability that it is from the first urn?

(a) $\frac{2}{5}$ (b) $\frac{2}{7}$ (c) $\frac{4}{7}$ (d) $\frac{4}{9}$

37. The probability of getting a defective floppy in three boxes A, B and C are 1/3, 1/6, and 3/4, respectively. A box is selected randomly and a floppy is drawn from it. The probability that the floppy is defective and is drawn from box A is

(a) 4/15 (b) 12/15 (c) 2/15 (d) 3/5

The probability density function of the random variable 'x' is given as

x:	0	1	2	3	4	5	6
$P(x)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

38. The probability that the value of 'x' will be less than 4, will be

(a) 9/49 (b) 16/49 (c) 24/49 (d) 25/49

39. The probability that the value of 'x' will be atleast '5', will be

(a) 9/49 (b) 16/49 (c) 24/49 (d) 25/49

40. An unbiased coin is tossed 6 times. What is the probability of obtaining 4 heads?

(a) 3/16 (b) 15/64 (c) 15/32 (d) 21/64

41. The overall percentage of failure in an examination is 20 percent. If 7 candidates appear in the examination, what is the probability that atleast 6 pass the examination?

(a) $\frac{4^6.11}{5^7}$ (b) $\frac{4^7.11}{5^7}$ (c) $\frac{4^7.11}{5^6}$ (d) $\frac{4^6.11}{5^6}$

42. A die is thrown 9 times. What is the probability of getting '5' exactly 3 times?

(a) $\frac{5^6.14}{6^9}$ (b) $\frac{5^6.14}{6^8}$ (c) $\frac{5^6.84}{6^8}$ (d) $\frac{5^3.84}{6^8}$

43. The chance that out of 10 telephone lines, one is busy at a time is 0.8. What will be the most probable number of busy lines?

(a) 2 (b) 5 (c) 8 (d) 10

44. Three unbiased coins are tossed together and this is repeated five times. What is the probabilty of getting atleast one head each time?

(a) $\frac{1}{8}$ (b) $\left(\frac{1}{8}\right)^5$ (c) $\frac{7}{8}$ (d) $\left(\frac{7}{8}\right)^5$

45. Which of the following is NOT an assumption of the Binomial distribution?
(a) All trials must be identical.
(b) All trials must be independent.
(c) Each trial must be classified as a success or a failure.
(d) The probability of success is equal to 0.5 in all trials.

46. In a triangle test, a tester is presented with three food samples, two of which are alike, and is asked to pick out the odd one by testing. If a tester has no well developed sense and can pick the odd one only, by chance, what is the probability that in five trials he will make four or more correct decisions?
(a) 11/243 (b) 1/243 (c) 10/243 (d) 233/243

47. The probability that a certain machine will produce a defective item is 1/4. If a random sample of 6 items is taken from the output of this machine, what is the probability that there will be 5 or more defectives in the sample?
(a) 3/4096 (b) 4/4096 (c) 18/4096 (d) 19/4096

48. Suppose 60% of a herd of cattle is infected with a particular disease. Let Y = the number of non-diseased cattle in a sample of size 5. The distribution of Y is
(a) binomial with n = 5 and p = 0.6
(b) binomial with n = 5 and p = 0.4
(c) binomial with n = 5 and p = 0.5
(d) none of these

49. The chance of a coal miner being killed in a mine accident is 1/1400 during a calendar year. What is probability that a mine with 3500 employers will have at least one accident in a year? (Given: $e^{-5/2} = 0.082$)

(a) 0.164 (b) 0.492 (c) 0. 618 (d) 0.918

50. There are 20 buses passing a bus-stop randomly within an hour. What is the probability that there will be no buses within 5 minutes? (Given: $e^{-5/3} = 0.19$)

(a) 0.19 (b) 0.23 (c) 0.38 (d) 0.52

51. A card is drawn from a well-shuffled pack of 52 cards. A sequence of 156 trials is made. What is the probability of getting a queen of clubs at least once?

(a) $\frac{1}{e^3}$ (b) $1-\frac{1}{e^3}$ (c) $1-\frac{1}{e^{52}}$ (d) $\frac{1}{e^{52}}$

52. Suppose flaws (cracks, chips, specks, etc.) occur on the surface of glass with density of 3 per square metre. What is the probability of there being exactly 4 flaws on a sheet of glass of area 0.5 square metre? (Given: $e^{-3/2} = 0.223$)
(a) 0.047 (b) 0.168 (c) 0.815 (d) 0.647

53. In a certain communications system, there is an average of 1 transmission error per 10 seconds. What is the probability of more than 1 error in a communication one-half minute in duration? (Given: $e^{-3} = 0.05$)
(a) 0.2 (b) 0.32 (c) 0.56 (d) 0.8

54. Students of a class were given an aptitude test. Their marks were found to be normally distributed with mean 60 and standard deviation 5. What percentage of students scored more than 60 marks?

(a) 0.25 (b) 0.5 (c) 0.6 (d) 0.75

55. In a test on 2000 electric bulbs, it was found that the life of a particular bulb was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. How many bulbs likely to burn for more than 2160 hours?

(a) 45 (b) 65 (c) 84 (d) 105

ANSWER KEY

1. (b) 2. (c) 3. (a) 4. (c) 5. (b) 6. (a) 7. (d)
8. (c) 9. (a) 10. (b) 11. (c) 12. (d) 13. (b) 14. (c)
15. (d) 16. (b) 17. (d) 18. (d) 19.(c) 20. (a) 21. (a)
22. (a) 23. (c) 24. (d) 25. (a) 26. (a) 27. (c) 28. (c)
29. (c) 30. (b) 31. (b) 32. (c) 33. (a) 34. (b) 35. (a)
36. (c) 37. (a) 38. (b) 39. (c) 40. (b) 41. (a) 42. (b)
43. (c) 44. (d) 45. (d) 46. (a) 47. (d) 48. (b) 49. (d)
50. (a) 51. (b) 52. ( ) 53. ( ) 54. (b) 55. (a)

Chapter 9

NUMERICAL ANALYSIS

9.1 Significant Figures

(1) All non-zero digits are significant Example: 123.45 has 5 significant figures.

(2) Zero's appearing between the non-zero's digits are significant. Example: 102.305 has 6 significant figures.

(3) Leading zero's are not significant. Example: 0.000465 has 3 significant figures.

(4) Trailing zero's are significant. Example: 1230.00 has 6 significant figures.

9.2 Rounding-off numbers

(1) If (n + 1)th digit is greater than 5, then nth digit will be incearsed by 1.

Example: $3.14765 \approx 3.148$

(2) If (n + 1)th digit is equal to 5 and it is followed by a non-zero digit, then nth digit will be increased by 1.

Example: $3.14751 \approx 3.148$

(3) If (n + 1)th digit is equal to 5 and it is followed by a zero digit then

(a) nth digit will be increased by 1, if it is odd

(b) nth digit will remain unchanged if it is even.

Example: $3.14750 \approx 3.148, 3.14650 \approx 3.146$

Round-off Error: $E_R = X$ (original value) – X_1 (Round off value)

Relative Error: $E_r = \dfrac{X - X_1}{X}$

Percentage Error: $E_p = \dfrac{X - X_1}{X} \times 100\%$

Maximum Round-off Error: $|E_R| \le 0.5 \times 10^{-n+1}$ ('n' is the decimal place upto which the given number should be rounded-off.

Example 1: Computing $\sqrt{1} + \sqrt{2} + \sqrt{3} + \dots\dots + \sqrt{100}$ with all roots correct to two decimal places, the maximum possible error in the result will be

(a) 0.1 (b) 0.01 (c) 0.5 (d) 5

Soln: Maximum round off error for a root of sinngle number upto two decimal place $= 0.5 \times 10^{-2+1} = 0.05$

Maximum round off error in the total evaluation $= 100 \times 0.05 = 5$

Example 2: Round-off the number 4.5126 to four significant figures and find the relative percentage error.

Soln: The fifth significant fugure is 6, thus the round-off number upto four significant figures will 4.513

Relative percentage error $= \dfrac{(4.5126 - 4.513)}{4.5126} \times 100\% = -0.0088\%$

9.3 Solution of Polynomial and Transcedental equations

Transcedental Equation:

If $f(x)$ contains both algebraic and other type of function then the equation is said to be transdental equation.

Example: $x = \tan x$, this type of equation is generally solved by graphical or numerical method.

Polynomial Equation:

If $f(x)$ contains only algebraic function of x, then the equation is polynomial equation

Example: $x^3 - 2x^2 + x + 6 = 0$

Note: (1) If $f(x)$ is continuous on $[a,b]$ and a,b are two real numbers such that

(a) $f(a)$ and $f(b)$ are of opposite signs, then there are odd number of real roots of the equation $f(x) = 0$ in (a,b).

(b) $f(a)$ and $f(b)$ are of same signs, then there are even number of real roots or no real root of the equation $f(x) = 0$ in (a,b).

(2) For any algebric eqution $f(x) = 0$

(a) The number of the positive roots cannot exceed the number of changes of signs of the co-efficients in the polynomial $f(x)$.

(b) The number of the negitive roots cannot exceed the number of changes of signs of the co-efficients in the polynomial $f(-x)$.

Example 2: If their is a real root of $x^3 - 3x + k = 0$ in $[-1, 1]$ then find the range of k.

Soln: $f(-1) = (-1)^3 - 3(-1) + k = k - 1 + 3 = k + 2$

$f(1) = (1) - 3(1) + k = k - 2 = k - 2$

Since, there is only one root, f(1) and f(-1) are of opposite sign

$k + 2 > 0 \Rightarrow k > -2$ and $k - 2 < 0 \Rightarrow k < 2$; therfore the range of k: $\boxed{-2 < k < 2}$

9.4 Bisection Method

This is a root finding algorithm, which repeated divides an interval into half and then selects the subinterval in which a root exists. Suppose we have two points a and b such that f(a) and f(b) has oppsite signs, then by roll's theorem f(x) must have atleast one root in the intetval [a,b]. In this method, the another point c is computed between a and b, such that $c=\frac{a+b}{2}$. Now there are two possibilities: (i) f(a) and f(c) have opposite signs or (ii) f(b) and f(c) have opposite signs. To get the desired accuracy, bisection method is then applied recursively to that sub-interval, where sign change exists.

Steps:

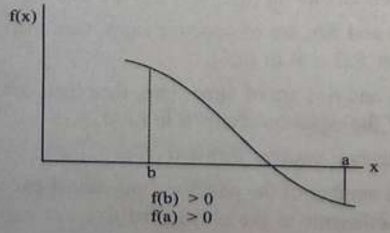
(1) Try to find out the interval $I_0=[a_0,b_0]$ such that $f(a_0)f(b_0)<0$. It means that the root is lying within the intetval I_0.

(2) Find the mid-point $a_1=\frac{a_0+b_0}{2}$.

(3) Now, select the next interval $I_1=[a_0,a_1]$ if $f(a_0)f(a_1)<0$ or $I_1=[a_1,b_0]$ if $f(a_1)f(b_0)<0$.

(4) Repeat this process to get I_2, I_3........If the procedure is repeated n times, either we find the root or the interval of length $\frac{b_0-a_0}{2^n}$ which contains the root.

It is clear that the error in the root neve be greater than half of the length of the interval of which it is the midpoint. If the permissible error is $\in$, then the number of iterations can be determined by $\frac{b_0-a_0}{2^n}\le\in$ i,e $n\ge\frac{\ln(b_0-a_0)-\ln\in}{\ln 2}$.



Example 3: Find the real roots of the equation (real) $x^3-2x-5=0$:

Soln: Now, $f(x)=x^3-2x-5\Rightarrow f(2)=-1$ and $f(3)=16$

hence a root lies between 2 and 3 and Initial guess: $x_0=\frac{2+3}{2}=2.5\Rightarrow f(x_0)=f(2.5)=5.6250=$ positive

Second Guess: $x_1=\frac{2+2.5}{2}=2.25\Rightarrow f(x_1)=f(2.25)=1.8=$ positive

Proceeding in this way the following table is obtained.

N	A	B	X	F(x)
1	2	3	2.5	5.6250
2	2	2.5	2.25	1.8906
3	2	2.25	2.125	0.3457
4	2	2.125	2.0625	-0.3513
5	2.0625	2.125	2.09375	-0.0089
6	2.09375	2.125	2.10938	0.1668
7	2.09375	2.10938	2.10156	0.07856
8	2.09375	2.10156	2.09766	0.03471
9	2.09375	2.09766	2.09570	0.01286
10	2.09375	2.09570	2.09473	0.00195
11	2.09375	2.09478	2.09424	-0.0035

Therefore, one of the roots of the above equation is $x = 2.09424$ (approx).

Example 4: Find the solution of the equation $x^3 - 4x - 9 = 0$

Soln: $f(x) = x^3 - 4x - 9$

Now, $f(3) = 21 - 12 - 9 = 6$ (+ve) and $f(2) = 8 - 8 - 9 = -9$ (–ve)

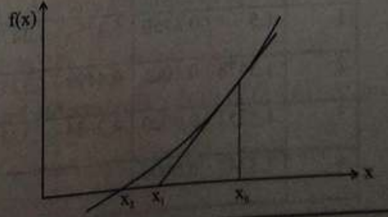
First guess value: $x_1=\frac{1}{2}(2+3)=2.5$ {mean of 2 and 3}; $f(2.5) = -3.375$ (–ve)

Second guess value: $x_2=\frac{1}{2}(2.5+3)=2.75\Rightarrow f(2.75)=0.7969$ (+ve)

Third guess value: $x_3=\frac{1}{2}(2.5+2.75)=2.625$ and this will go so on until $f(x) = 0$ for some guess value. This method is not so accurate.

9.5 Newton-Raphson Method

Let x_0 be a point near the root α of the equation $f(x) = 0$, then tangent to the curve at $A\{x_0, f(x_0)\}$ is $y-f(x_0)=f'(x_0)(x-x_0)$. It cuts the X-axis at $x_1=x_0-\frac{f(x_0)}{f'(x_0)}$ which is first approximation to root α. If A_1 correponds to x_1 on the curve, then tangent to the curve at A_1 will cut X-axis at x_2, nearer to α and is therefore second approximation to root α. Repeating this process, we approach to the exact root α.



Steps:

(i) Choose two close numbers b and c such that $f(b)$ and $f(c)$ have oppostie sign. therefore the root lies between b and c.

(ii) The number out of $f(b)$ and $f(c)$, which nearest to zero, will be the first approximate root of the solution.

(iii) Now apply newton raphson formula for to a better approximate root:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example 5: Find the root of the equation $x^3 - 5x + 3 = 0$

Soln: $f(0) = 3$ (+ve)and $f(1) = -1$(–ve) i.e. there is one root is between 0 and 1.

First guess value: $x_1 = 1 \Rightarrow f(1) = -1$

Second guess value: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{2} = 0.5$

Third guess value: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.6471$

The consecutive value of x for which are nearlly equal, will be the root of the equation by this method.

Example 6: Calculate the Newton Raphson iterative formula to find $\sqrt{N}$.

Soln: $x = \sqrt{N} \Rightarrow x^2 - N = 0$

$f(x_n) = x_n^2 - N \Rightarrow f'(x_n) = 2x_n$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Example 7: Find the x co-orddinate of the point where the curve $y = x^3 - x$ cross the line $y = 1$.

Soln. We have to solve $x^3 - x - 1 = 0$

$f(x) = x^3 - x - 1 \Rightarrow f'(x) = 3x^2 - 1$

Then itaration formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Let $x_0 = 1$ the successive ittrates are given below

n	x_0	$f(x_n)$	$f'(x_n)$	X_{n+1}
0	1	–1	2	1.5
1	1.5	0.8750	5.375	1.3478
2	1.3478	0.1006	4.4499	1.3252
3	1.3252	0.0020	4.2684	1.3247
4	1.3247	0.0000		

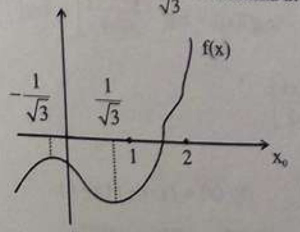
So, the root of the equation is 1.3247 (approx.)

Discussion:

$f(x) = x^3 - x - 1 \Rightarrow f(1) = -ve$ and $f(2) = +ve$

$f'(x) = 3x^2 - 1$; so, $f'(x) = 0$ at $x = \pm 1/\sqrt{3}$

$f''(x) = 6x$; therefore $f(x)$ has maxima at $x = -\frac{1}{\sqrt{3}}$ and minima at $x = \frac{1}{\sqrt{3}}$ – min



(1) If we start x_0 between $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ there can be trouble.

(2) But if we can start any trace to the right of $x_0 = \frac{1}{\sqrt{3}}$ then there is no problem.

(3) Newton method is stop if $f'(x_n)$ is zero.

This method is very accurate when the value of $f'(x)$ is very large

If $f'(x) = 0$ or nearly 0 then the method fails.

The method is also used to obtain complex roots.

9.6 Interpolation and Extrapolation

Let us consider the function y = f(x) for a set of values of x as

$$\begin{array}{llllll} x: & x_0 & x_1 & x_2 & \cdots\cdots & x_n \\ y: & y_0 & y_1 & y_2 & \cdots\cdots & y_n \end{array}$$

The method of finding y corresponding to any value of x between x_0 to x_n, is known as interpolation. The method to calculate y corresponding to value of x outside the range from x_0 to x_n is known as extrapolation.

Forward difference:

(1) First order forward difference:

$\Delta f(x) = f(x+h) - f(x)$ (Δ = forward difference operator)

$\Delta y_0 = y_1 - y_0,\ \Delta y_1 = y_2 - y_1 \ \cdots\cdots$

(2) Second order forward difference: $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$

Similarly, $\Delta^2 y_1 = y_3 - 2y_2 + y_1$

In general, p^th forward difference: $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$

Example 8: Calculate $\Delta \log[f(x)]$

Soln: $\Delta \log f(x) = \log[f(x+h)] - \log[f(x)] = \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$

$\Rightarrow \Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Backward Difference:

$$\nabla f(x) = f(x) - f(x-h)$$

In general, p^th backward diference: $\nabla^p y_r = \nabla^{p-1} y_r - \nabla^{p-1} y_{r-1}$

Central Difference:

$$\delta^p y_r = \delta^{p-1} y_{r+\frac{1}{2}} - \delta^{p-1} y_{r-\frac{1}{2}}$$

Shift Operator:

$$Ef(x) = f(x+h)$$

Similarly, $E^2 f(x) = f(x+2h)$, $E^n f(x) = f(x+nh)$

Average Operator:

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

Relation between Δ and E operator:

$\Delta f(x) = f(x+h) - f(x) \Rightarrow \Delta f(x) = Ef(x) - f(x) \Rightarrow \Delta f(x) = (E-1)f(x) \Rightarrow \Delta = E - 1$

Relation between ∇ and E operator:

$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x) \Rightarrow \nabla = 1 - E^{-1}$

Newton Interpolation Formula:

(1) The interval between two consecutive values of x should be equal.

(2) The given y or $f(x)$ should be a increasing or decreasing function of x.

(1) **Forward Interpolation formula:**

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \text{ where } u = \frac{x-x_0}{h}$$

---

(2) **Backward interpolation formula:**

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \ldots\ldots\ldots\ldots\ldots$$

Exanple 9: Given:

x:	1	2	3	4
$f(x)$:	1	7	17	31

Find the value of $f(1.5)$.

Soln:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1	6	4	0
2	7	10	4	
3	17	14		
4	31			

$u = \frac{1.5-1}{1} = 0.5$; $f(1.5) = 1 + 0.5(6) + \frac{(0.5)(0.5-1)}{2!}.(4) \Rightarrow f(1.5) = 3.5$

Example 10: Find the polynomial corresponding to the data given in the previuos question:

Soln: $u = \frac{x-x_0}{h} = \frac{x-1}{h} = (x-1)$

$y = f(x) = 1 + (x-1).6 + \frac{(x-1)(x-2)}{2!}.4 = 1 + 6x - 6 + 2x^2 + 4 - 6x = 2x^2 - 1$

The order upto which the forward difference are non-zero, that will be the order of the polynomial corresponding to the given data.

Example 11: The following describes a 4th order polynomial. What will be the missing terms:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0	3	-6	12	-24	$y_1 + 45$
1	3	-3	6	-12	$y_1 + 21$	$y_2 - 5y_1 - 33$
2	0	3	-6	$y_1 + 9$	$y_2 - 4y_1 - 12$	
3	3	-3	$y_1 + 3$	$y_2 - 3y_1 - 3$		
4	0	y_1	$y_2 - 2y_1$			
5	y_1	$y_2 - y_1$				
6	y_2					

Since, the given data correcponds to fourth order polynomial, therefore the fifth order difference will be zero i.e. $y_1 + 45 = 0 \Rightarrow y_1 = -45$

$y_2 - 5(-45) - 33 \Rightarrow y_2 = -192$

9.7 Numerical Integration

Evaluation of $\int_{x_0}^{x_n} f(x)dx$

For this, divide the limits into interval: x: x_0 x_1 x_2 .... x_n

$f(x)$: y_0 y_1 y_2 .... y_n

Trapezoidal Rule:

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots y_{n-1})] \quad \text{where } h = x_1 - x_0$$

Simpson's 1/3rd Rule: (for even no. of intervals)

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]$$

Simpson's 3/8th rule: (for number of intervals multiple of 3)

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots)\right]$$

Example 13:

x :	−1	0	1	2	3
$f(x)$:	−1	5	3	1	5

Calculate: $\int_{-1}^{3} f(x)dx = ?$

Soln: Trapezoidal rule: $\int_{-1}^{3} f(x)dx = \frac{1}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)] = 12$

Simpson 1/3rd rule: $\int_{-1}^{3} f(x)dx = \frac{1}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)] = 12$

9.8 Numerical Solution of Differential equation

RUNGE KUTTA METHOD

$\frac{dy}{dx} = f(x, y)$ Given: $x = x_0, y = y_0$; Find the value of y at $x = x_0 + \Delta x$.

Suppose at $x = x_0 + \Delta x$, $y = y_0 + \Delta y$

Assuming, $\Delta x = h$ (say) $\Delta y = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}$

where, $K_1 = hf(x_0, y_0)$, $K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$,

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$, $K_4 = hf(x_0 + h, y_0 + K_3)$

Example 14: $\frac{dy}{dx} = (x + y)$ given y(0) = 1, find y(0.2).

Soln: $x_0 = 0$ and $(x_0 + \Delta x) = 0.2 \Rightarrow \Delta x = 0.2 = h$

$K_1 = hf(x_0, y_0) = 0.2(0 + 1) = 0.2$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2(0.1 + 1.1) = 0.2 \times 1.2 = 0.24$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2(0.1 + 1.12) = 0.2 \times 1.22 = 0.944$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.2(0.2 + 1.244) = 0.2 \times 1.444 = 0.2888$$

$$\Delta y = \frac{0.2 + 2(0.24) + 2(0.244) + 0.2888}{6} = \frac{0.2 + 0.48 + 0.488 + 0.2888}{6}$$

Therefore, $y(0.2) = y_0 + \Delta y$

Example 15: $\frac{dy}{dx} = 1 + y^2$ given that $y = 0$ at $x = 0$; Find y(0.2).

Soln: $h = 0.2$ $\quad f(x) = 1 + y^2$

$k_1 = (0.2).1 = 0.2$

$k_2 = 0.2 \times f(0 + 0.1, 0 + 0.1) = 0.2 \times f(0.1, 0.1) = 0.202$

$k_3 = 0.2 \times f(0 + 0.1, 0 + 0.101) = 0.2 \times f(0.1, 0.101) = 0.2020402$

$k_4 = 0.2 \times f(0 + 0.2, 0 + 0.2020402) = 0.2 \times f(0.2, 0.2020402) = 0.20816$

$$y(0.2) = 0 + \frac{1}{6}[0.2 + 0.404 + 0.40408 + 0.20816] = \frac{1}{6}[1.2161] = 0.2027$$

PRACTICE SET

1. Each of the following numbers are rounded-off upto three decimal places:

$$3.14750,\ 10.54761,\ 8.23450,\ 3.46756$$

Which of the following is true?

(a) $3.14750 \approx 3.148,\ 10.54761 \approx 10.548,\ 8.23450 \approx 8.235,\ 3.46756 \approx 3.468$

(b) $3.14750 \approx 3.148,\ 10.54761 \approx 10.548,\ 8.23450 \approx 8.234,\ 3.46756 \approx 3.468$

(c) $3.14750 \approx 3.147,\ 10.54761 \approx 10.548,\ 8.23450 \approx 8.234,\ 3.46756 \approx 3.468$

(d) $3.14750 \approx 3.148,\ 10.54761 \approx 10.547,\ 8.23450 \approx 8.234,\ 3.46756 \approx 3.468$

2. If the number 9.0894 is rounded-off to three significant figures, then the round-off error is

(a) 0.0004 (b) -0.0004 (c) 0.001 (d) -0.001

3. Consider the following series:

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\dots\dots\dots+\sqrt{100}$$

If each of the roots are calculated upto rounding-off two decimal places, then the maximum possible error in the calculation of the sum of the series, is

(a) 0.05 (b) 0.5 (c) 5 (d) 50

4. The interval for the smallest positive root of the transcendental equation $x = e^x$, is

(a) [-1,0] (b) [-1,1] (c) [0,1] (d) [1,2]

5. If the equation $x^3 - 3x + K = 0$ has a real root in the interval [-1,1], the the range of K is

(a) (-2,2) (b) (0,2) (c) (-2,0) (d) (1,3)

6. The Newton raphson iterative formula to evaluate the square root of a natural number 'N' is

(a) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ (b) $x_{n+1} = \frac{1}{2}\left(x_n - \frac{N}{x_n}\right)$

(c) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{2x_n}\right)$ (d) $x_{n+1} = \frac{1}{2}\left(x_n - \frac{N}{2x_n}\right)$

7. The Newton raphson iterative formula to evaluate the cube root of a natural number 'N' is

(a) $x_{n+1} = \frac{3}{2}\left(x_n + \frac{N}{2x_n^2}\right)$ (b) $x_{n+1} = \frac{3}{2}\left(x_n - \frac{N}{2x_n^2}\right)$

(c) $x_{n+1} = \frac{2}{3}\left(x_n + \frac{N}{2x_n^2}\right)$ (d) $x_{n+1} = \frac{2}{3}\left(x_n - \frac{N}{2x_n^2}\right)$

8. Consider the following equation: $x^4 - x - 10 = 0$.
To solve the equation numerically using Newton-Raphson method, which one will be the best possible first guess value?

(a) 0 (b) 1 (c) 3/2 (d) 2

9. The first order forward difference of $\ln\left[f(x)\right]$ is

(a) $\ln\left[1+\frac{\Delta f(x)}{f(x)}\right]$ (b) $\ln\left[1-\frac{\Delta f(x)}{f(x)}\right]$ (c) $\ln\left[1-\frac{f(x)}{\Delta f(x)}\right]$ (d) $\ln\left[1+\frac{f(x)}{\Delta f(x)}\right]$

10. The second order forward difference of $f(x) = \cos 2x$ is

(a) $4\sin^2 h.\cos(2x+2h)$ (b) $-4\sin^2 h.\cos(2x+2h)$

(c) $-4\cos^2 h.\sin(2x+2h)$ (d) $4\cos^2 h.\sin(2x+2h)$

Consider the following data:

x:	1	2	3	4
$f(x)$:	1	7	17	31

11. The value of $f(x)$ at x =1.5 is

(a) 2.5 (b) 3 (c) 3.5 (d) 4

12. The functional relationship between x and $f(x)$ for the above mentioned data, is

(a) $f(x) = x^2 - 1$ (b) $f(x) = 2x^2 - 1$ (c) $f(x) = 2x^2 + 1$ (d) $f(x) = x^2 + 1$

13. Consider the following data:

x:	0	1	2	3
$f(x)$:	0	1	3	10

The polynomial corresponding to the given data, is

(a) $\frac{2}{3}x^3 - \frac{3}{2}x^2 + \frac{11}{6}x$ (b) $\frac{2}{3}x^3 - \frac{3}{2}x^2 - \frac{11}{6}x$ (c) $\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{11}{6}x$ (d) $\frac{2}{3}x^3 + \frac{3}{2}x^2 + \frac{11}{6}x$

14. Consider the following data:

x:	0	1	2	3	4	5	6
$f(x)$:	0	3	0	3	0	?	?

If the given data a describes fourth order polynomial, then the missing terms are

(a) 45,192 (b) 45,−192 (c) −45,192 (d) −45,−192

15. Consider the following operators:

$$E\left[f(x)\right] = f(x+h);\ D\left[f(x)\right] = \frac{df}{dx}$$

Which of the following relation is true?

(a) $E = e^{hD}$ (b) $E = e^{-hD}$ (c) $E = 1 + e^{hD}$ (d) $E = 1 - e^{hD}$

16. Consider the following opeators:

$$\delta\left[f(x)\right] = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right);\ \Delta\left[f(x)\right] = f(x+h) - f(x)$$

Which of the following relations is true?

(a) $\Delta = \frac{1}{2}\delta^2 - \delta\sqrt{1+\frac{1}{4}\delta^2}$ (b) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{1}{4}\delta^2}$

(c) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1-\frac{1}{4}\delta^2}$ (d) $\Delta = \frac{1}{2}\delta^2 - \delta\sqrt{1-\frac{1}{4}\delta^2}$

17. The missing term in the following table:

x:	2	3	4	5	6
$f(x)$:	45.0	49.2	54.1	?	67.4

(a) 12.5 (b) 24.75 (c) 45.8 (d) 60.05

18. The missing values of the following table:

x:	45	50	55	60	65
$f(x)$:	3.0	?	2.0	?	−2.4

(a) 2.925,0.225 (b) 2.525, 0.225 (c) 2.925, 0.775 (d) 2.525, 0.775

19. Consider the following data:

x:	−1	0	1	2	3
$f(x)$:	1	5	3	1	5

The value of the integral $\int_{-1}^{3} f(x)dx$ is

(a) 8 (b) 10 (c) 12 (d) 16

20. Consider the following data:

x:	3	6	9	12
$f(x)$:	−1	1	2	3

The value of the integral $\int_{3}^{12} f(x)dx$ is

(a) 15 (b) 10 (c) 12 (d) 5

21. To evaluate $\int_{x_0}^{x_n} ydx$ by Simpson's 1/3 rd rule as well as Simpson's 3/8 th rule, the number of intervals should be

(a) 2 (b) 3 (c) 6 (d) 10

22. To evaluate $\int_{x_0}^{x_n} ydx$ by Simpson's 1/3 rd rule as well as Trapezoidal rule, the number of intervals should be

(a) 2 (b) 3 (c) 6 (d) 10

A second degree polynomial f(x) has the following data:

x:	0	1	2
$f(x)$:	1	4	15

23. The value of the integral $\int_{0}^{2} f(x)dx$ (using Trapezoidal rule) will be

(a) 8 (b) 12 (c) 16 (d) 20

24. The error in the estimation by trapezoidal rule, will be

(a) -4/3 (b) 4/3 (c) 5/3 (d) -5/3

25. Consider the following differential equation: $\frac{dy}{dx} = -y$ given that, at $x = 0,\ y = 1$

The value of y at x = 0.1 (using Runge-kutta method of third order) is

(a) 0.24 (b) 0.46 (c) 0.91 (d) 0.99

26. Consider the following differential equation: $\frac{dy}{dx} = x + y$ given that, at $x = 0, y = 1$

The value of y at x = 0.2 (using Runge-kutta method of fourth order) is

(a) 0.94 (b) 1.11 (c) 1.24 (d) 1.67

ANSWER KEY

1. (b) **2. (d)** **3. (c)** **4. (c)** **5. (a)** **6. (a)** **7. (c)**
8. (d) **9. (a)** **10. (b)** **11. (c)** **12. (b)** **13. (a)** **14. (d)**
15. (a) **16. (b)** **17. (d)** **18. (a)** **19. (c)** **20. (c)** **21. (c)**
22. (a) **23. (b)** **24. (a)** **25. (c)** **26. (c)**

Chapter 10

TENSOR ANALYSIS

10.1 Basic Review of Tensors

Tensors are a logical generalization of vectors. It is well-known that the use of vectors is very much essential in the study of various physical phenomena. Similarly, tensors has many applications in various branches in physics. These applications can be divided into two sections: (a) application in non-relativistiv physics and (ii) appication in general theory of Relativity.

According to Ohm's Law i.e the electric current in medium is proportional to the aplied electric field that can be mathematically written as $\vec{J} = \sigma\vec{E}$, where $\vec{J}$ is the electric current, σ is the electrical conductivity and $\vec{E}$ is the applied electri field. This mathematical form of Ohm's law is strictly applicable for isotropic medium. But in many anaisotropic medium, current flows in directions that are quite different from the applied electric field. The generalized form of above equation can be written as

$$J_x = \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z$$

$$J_y = \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z$$

$$J_z = \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z$$

where J_x, J_y, J_z and E_x, E_y, E_z are the cartesian components $\vec{J}$ and $\vec{E}$, σ_{ij} is the components of the electrical conductivity tensor of the medium.

Tensors are asscociated with magnitude and two or more directions.

Example: (i) Stress of an elastic solid at a point depends upon two directions:

(a) Direction of the area, (b) direction of the force

(ii) Electrical conductivity of the medium is associated with two directions:

(a) Direction of the electric field, (b) Direction of flow of electric current

Einstein Summation Convention:

"Repeated indices are summed over"

Consider an N-dimensional space and $x^1, x^2, \ldots\ldots x^N$ are a set of coordinates in this space. It is important to note that $x^1, x^2, \ldots\ldots x^N$ are not powers of x.

Example: (i) $a_i x^i = a_1 x^1 + a_2 x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n$

(ii) If $f = f\left(x^1, x^2, \ldots\ldots x^n\right)$, then,

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \ldots\ldots\ldots\ldots + \frac{\partial f}{\partial x^n}dx^n = \frac{\partial f}{\partial x^i}dx^i$$

(iii) $\dfrac{\partial v_i}{\partial x_i} = \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3}$

(v) $\dfrac{\partial \phi}{\partial x_i \partial x_i} = \dfrac{\partial^2 \phi}{\partial x_1^2} + \dfrac{\partial^2 \phi}{\partial x_2^2} + \dfrac{\partial^2 \phi}{\partial x_3^2}$

Contravariant tensor:

If the components of a tensor A tranforms between the (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$ co-ordinate system according to the following relation:

$$\bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^i} A^i$$

then, it is said to be contravariant tensor of rank 1.

In general, for Contravariant tensor of rank p

$$\bar{A}^{\mu_1\mu_2\ldots\ldots\mu_p} = \frac{\partial \bar{x}^{\mu_1}}{\partial x^{\alpha_1}} \frac{\partial \bar{x}^{\mu_2}}{\partial x^{\alpha_2}} \frac{\partial \bar{x}^{\mu_3}}{\partial x^{\alpha_3}} \cdots \frac{\partial \bar{x}^{\mu_p}}{\partial x^{\alpha_p}} A^{\alpha_1\alpha_2\ldots\ldots\alpha_p}$$

Example: Tangent vector, Velocity vector are contravariant tensors of rank 1.

Covariant tensor:

If the components of a tensor A tranforms between the (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$ co-ordinate system according to the following relation:

$$\bar{A}_j = \frac{\partial x^i}{\partial \bar{x}^j} A_i$$

then, it is said to be covariant tensor of rank 1.

In general, for Covariant tensor of rank q

$$\bar{A}_{\mu_1\mu_2\ldots\ldots\mu_q} = \frac{\partial x^{\alpha_1}}{\partial \bar{x}^{\mu_1}} \frac{\partial x^{\alpha_2}}{\partial \bar{x}^{\mu_2}} \frac{\partial x^{\alpha_3}}{\partial \bar{x}^{\mu_3}} \cdots \frac{\partial x^{\alpha_q}}{\partial \bar{x}^{\mu_q}} A_{\alpha_1\alpha_2\ldots\alpha_q}$$

Example: Normal vector to a surface is a covariant tensor of rank 1.

Example 1: A covariant tensor has components $2x - z$, x^2y and yz in Cartesian coordinate system. Find its components in cylindrical coordinate system.

Soln: $A_x = 2x - z = 2\rho\cos\phi - z$

$$A_Y = \rho^2\cos^2\phi(\rho\sin\phi) = \rho^3\cos^2\phi\sin\phi$$

$$A_z = yz = \rho z\sin\phi$$

$$\overline{A}_x = \frac{\partial x^1}{\partial \overline{x}^1} A_x + \frac{\partial x^2}{\partial \overline{x}^1} A_y + \frac{\partial x^3}{\partial \overline{x}^1} A_z$$

[Einstein convention: $x^1 = x, x^2 = y, x^3 = z, \overline{x}^1 = \rho, \overline{x}^2 = \phi, \overline{x}^3 = z$]

$$\Rightarrow \overline{A}_x = \frac{\partial x}{\partial \rho} A_x + \frac{\partial y}{\partial \rho} A_y + \frac{\partial z}{\partial \rho} A_z = \cos\phi[2\rho\cos\phi - z] + \sin\phi[\rho^3 \cos^2\phi \sin\phi]$$

$$\Rightarrow \overline{A}_x = 2\rho\cos^2\phi - 2\cos\phi + \rho^3\cos^2\phi\sin^2\phi$$

$$\overline{A}_y = \frac{\partial x^1}{\partial \overline{x}^2} A_x + \frac{\partial x^2}{\partial \overline{x}^2} A_y + \frac{\partial x^3}{\partial \overline{x}^2} A_z = \frac{\partial x}{\partial \phi} A_x + \frac{\partial y}{\partial \phi} A_y + \frac{\partial z}{\partial \phi} A_z$$

$$= -\rho\sin\phi(2\rho\cos\phi - z) + \rho\cos\phi(\rho^3\cos^2\phi\sin\phi) + 0$$

$\Rightarrow \overline{A}_y = -\rho^2 \sin 2\phi + \rho z \sin\phi + \rho^4 \cos^3\phi \sin\phi$; Similarly, $\overline{A}_x = \rho z \sin\phi$

Symmetric Tensor:

A tensor is said to be symmetric tensor if the component satisfies the relation: $A_{mn} = A_{nm}$

Note: A symmetric tensor A_{mn} has $\frac{n(n+1)}{2}$ independent components in n dimension.

Antisymmetric Tensor:

A tensor is said to be anti-symmetric tensor if the component satisfies the relation: $A_{mn} = -A_{nm}$

Note: An anti-symmetric tensor A_{mn} has $\frac{n(n-1)}{2}$ independent components in n dimension.

10.2 Alzebra of Tensors

Tensors of same type:

If two tensors have equal number of contravariant and covariant indices, then they are said to be of same type.

Example: A^i_{jk} and B^m_{np} are tensors of same type.

Note: Addition and substraction of tensors is possible only if they are of same rank and same type.

$A^i_{jk}(\pm)B^m_{mp} = C^x_{yz}$ possible ; $A^i_{jk}(\pm)B^{np}_m \neq C^x_{yz}$ not possible

Outer Product of tensors:

If each component of one tensor rank m, is mutiplied with every component of other tensor of rank n, then we will get tensor of rank $(m + n)$. This is known as outer product of tensors.

Example: $A^i_{jk}\, B^m_{np} = C^{im}_{jknp}$

Contraction of Tensor:

The process of making one contravariant index and one covariant index equal, is known as contraction of tensor. By a single contraction process the rank of the tensor get decreased by two.

Example: $A^{ij}_{mnp} \xrightarrow{i=m} A^j_{np}$

Inner Product of Tensors:

It can be thought of a outer product of two tensors followed by a single contraction process.

Example: $A^i_{jk}\, B^j_{np} = C^i_{knp}$, $A^i_{jk}\, B^{kn}_p = C^{in}_{jp}$

Levi-Civita Tensor:

It is a covariant tensor of rank 3 and its components are mathematically denoted by ε_{ijk} such that

$$\varepsilon_{ijk} = 0 \quad \text{if any two indices are identical}$$
$$= +1 \quad \text{for even permutation of } i, j, k$$
$$= -1 \quad \text{for odd permutation of } i, j, k$$

Note: (i) There are 27 components of the tensor, out of which

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

and all other elements are non-zero.

(ii) It is an anti-symmetric tensor.

(iii) $\varepsilon_{ijk}\varepsilon^{mnp} = \begin{vmatrix} \delta^m_i & \delta^n_i & \delta^p_i \\ \delta^m_j & \delta^n_j & \delta^p_j \\ \delta^m_k & \delta^n_k & \delta^p_k \end{vmatrix}$

(iv) $\varepsilon_{ijk}\varepsilon^{inp} = \begin{vmatrix} \delta^n_j & \delta^p_j \\ \delta^n_k & \delta^p_k \end{vmatrix}$

(v) $\varepsilon_{ijk}\varepsilon^{ijp} = 2\delta^p_k$

(vi) $\varepsilon_{ijk}\varepsilon^{ijk} = 6$

Kronecker Delta Tensor:

We know that,

$$\delta^i_j = 1 \quad \text{for } i = j$$
$$= 0 \quad \text{for } i \neq j$$

In tensor notation, $\delta^i_j = \frac{\delta x^i}{\delta x^j}$

Example 2: Prove that, $A^{\alpha}B_{\alpha}$ is a scalar

Proof: $A'^{\mu}B'_{\mu} = A^{\alpha}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}B_{\beta}\frac{\partial x^{\beta}}{\partial x'^{\mu}} = A^{\alpha}B_{\beta}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\mu}} = A^{\alpha}B_{\beta}\frac{\partial x^{\beta}}{\partial x'^{\alpha}}$

$$= A^{\alpha}B_{\beta}\ \delta^{3}_{\alpha} = A^{\alpha}B_{\alpha} = A^{\mu}B_{\mu}$$

$\Rightarrow A^{\mu}B_{\mu}$ remains unchanged by thensor law of transformation and hence $A^{\mu}B_{\mu}$ is a scalar or invariant.

10.3 Fundamental Tensor

Let x^i and $x^i + dx^i$ are co-ordinates of two points in an N-dimensional space with resprct to some co-ordinate system. Distance between two points ds can be written as

$$(ds)^2 = g_{ij}dx^i dx^j = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{12}dx^1dx^2 +$$

$$g_{21}dx^2dx^1 + g_{13}dx^1dx^3 + g_{31}dx^3dx^1 + g_{23}dx^2dx^3 + g_{32}dx^3dx^2$$

where g_{ij} is known as the fundamental tensor or metric tensor corresponding to that particular co-ordinate system.

Properties of Metric Tenor:

(i) It covariant tensor of rank 2.

(ii) It is a function of x^i.

(iii) It is a symmetric tensor.

Fundamnetal Tensor in Cartesian co-ordinates:

Distance ds between the points (x, y, z) and $(x + dx, y + dy, z + dz)$ in 3-D cartesian co-ordinate system can be written as

$$(ds^2) = (dx)^2 + (dy)^2 + (dz)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Comparing with equation standard expression for the metric tensor, we get metric tensor for cartesian co-ordinate system as follows:

$$(g_{ij})_{cartesian} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fundamnetal Tensor in Spherical polar co-ordinates:

Distance ds between the points (r, θ, φ) and $(r + dr, \theta + d\theta, \varphi + d\varphi)$ in 3-D spherical polar co-ordinate system can be written as

$$(ds^2) = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\varphi)^2 = (dx^1)^2 + r^2(dx^2)^2 + r^2\sin^2\theta(dx^3)^2$$

Comparing with equation standard expression for the metric tensor, we get metric tensor for spherical polar co-ordinate system as follows:

$$(g_{ij})_{\text{spherical polar}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g^{ij} = [g_{ij}]^{-1} = \frac{Adj[g_{ij}]}{|g_{ij}|} = \frac{1}{r^4\sin^2\theta}\begin{pmatrix} r^4\sin^2\theta & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & r^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Example 3: In a co-ordinate system, the line element 'ds' is defined as:

$$(ds)^2 = (dr)^2 + r^2\sin\theta(d\theta)^2 + r\sin^2\phi(d\phi)^2 + r\sin 2\phi(dr d\phi)$$

Calculate the form of the metric tensor g_{ij}.

Soln: Comparing with equation standard expression for the metric tensor, we get metric tensor as follows:

$$g_{ij} = \begin{pmatrix} 1 & r\sin\phi\cos\phi & 0 \\ r\sin\phi\cos\phi & r^2\sin\theta & 0 \\ 0 & 0 & r\sin^2\phi \end{pmatrix}$$

Example 4: Find the determinant of the metric tensor corresponding to following line element

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3$$

Soln: Comparing with equation standard expression for the metric tensor, we get metric tensor as follows:

$$g_{ij} = \begin{pmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

Therefore, $|g_{ij}| = \begin{vmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 4$

Example 5: Write the law of transformation for the tensors (a) A^i_{jk}, (b) B^{mn}_{ijk}, (c) C^m

Soln: (a) $\bar{A}^p_{qr} = \frac{\partial \bar{x}p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial x^k}{\partial \bar{x}^r}A^i_{jk}$ (b) $\bar{B}^{pq}_{rst} = \frac{\partial \bar{x}^p}{\partial x^m}\frac{\partial \bar{x}^q}{\partial x^n}\frac{\partial x^i}{\partial \bar{x}^r}\frac{\partial x^j}{\partial \bar{x}^s}\frac{\partial x^k}{\partial \bar{x}^t}B^{mn}_{ijk}$ (c) $\bar{C}^p = \frac{\partial \bar{x}^p}{\partial x^m}C^m$

Example 6: Show that $\frac{\partial A_p}{\partial x^p}$ is not a tensor even though A_p is a covariant tensor of rank one.

Soln: By hypothesis, $\bar{A}_j = \frac{\partial x^p}{\partial \bar{x}^j}A_p$. Differentiating with respect to $\bar{x}^k$.

$$\frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial A^p}{\partial \bar{x}^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} A_p$$

$$= \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial A_p}{\partial x^q}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k\,\partial \bar{x}^j} A_p = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial A_p}{\partial x^q} + \frac{\partial^2 x^p}{\partial \bar{x}^k\,\partial \bar{x}^j} A_p$$

Since the second term on the right is present, $\frac{\partial A_p}{\partial x^q}$ does not transform as a tensor should.

Example 7: Consider a co-ordinate system (u, v, w) which is related to the cartesian co-ordinates by

$$x = vw,\ y = uw,\ z = uv$$

Find the metric tensor of the co-ordinate system (u, v, w).

Soln: The line element ds is as following:

$$(ds)^2 = \left(v^2 + w^2\right)du^2 + \left(u^2 + w^2\right)dv^2 + \left(u^2 + v^2\right)dw^2 + 2uvdudv + 2vwdvdw + 2uwdudw$$

Comparing with equation standard expression for the metric tensor, we get metric tensor as follows:

$$\left[g_{ij}\right] = \begin{bmatrix} v^2 + w^2 & uv & uw \\ uv & u^2 + w^2 & vw \\ uw & vw & u^2 + v^2 \end{bmatrix}$$

Associate Tensors:

$$A^i g_{ij} = A_j$$

A^i and A_j can be thought as the contravariant and covariant components of the same vector. Therefore, A^i and A_j are called associate tensors.

Raising and Lowering of indices:

(A) Contravariant to Covariant: $A^{ij} g_{im} g_{jn} = A_{mn}$

(B) Covariant to Contravariant: $A_{ij} g^{im} g^{jn} = A^{mn}$

10.4 Chrystoffel Symbols

These connect the properties at two points in curved space.

Chrystoffel symbol of 1st kind: $[ij, k] = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right]$

Chrystoffel Symbol of 2nd kind: $\Gamma^m_{ij} = \begin{Bmatrix} m \\ ij \end{Bmatrix} = g^{km}[ij, k]$

Example 8: Given that the metric tensor in spherical polar coordinate system

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Calculate $[2\,2, 3]$, $[2\,2, 1]$, Γ^1_{22}.

Soln: $[22, 3] = \frac{1}{2}\left[\frac{\partial g_{23}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^3}\right] = \frac{1}{2}\left[0 + 0 - \frac{\partial (r^2)}{\partial \phi}\right] = 0$

$[22, 1] = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right] = \frac{1}{2}\left[0 + 0 - \frac{\partial r^2}{\partial r}\right] = \frac{1}{2}(-2r) = -r$

$\Gamma^1_{22} = g^{11}[22,1] + g^{12}[22,1] + g^{13}[22,1] = 1[22,1] = -r$

Example 9: Show that $[ik, j] + [jk, i] = \frac{\partial g_{ij}}{\partial x^k}$

Soln: L.H.S. $= \frac{1}{2}\left[\frac{\partial . g^{ij}}{\partial x^k} + \frac{\partial g^{kj}}{\partial x^i} - \frac{\partial g^{ik}}{\partial x^j}\right] + \frac{1}{2}\left[\frac{\partial g^{ji}}{\partial x^k} + \frac{\partial g^{ki}}{\partial x^j} - \frac{\partial g^{jk}}{\partial x^i}\right] = \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k} + \frac{1}{2}\frac{\partial g^{ji}}{\partial x^k} = \frac{\partial g^{ij}}{\partial x^k}$

Example 10: Find g^{ij} if $(ds)^2 = (dx^1)^2 + 2\cos\xi dx^1 dx^2 + (dx^2)^2$

Soln: $g_{ij} = \begin{pmatrix} 1 & \cos\xi \\ \cos\xi & 1 \end{pmatrix}$, $g^{ij} = \begin{pmatrix} \frac{1}{\sin^2\xi} & \frac{-\cos\xi}{\sin^2\xi} \\ \frac{-\cos\xi}{\sin^2\xi} & \frac{1}{\sin^2\xi} \end{pmatrix} \Rightarrow g^{ij} = \begin{pmatrix} 1 & -\cos\xi \\ -\cos\xi & 1 \end{pmatrix}\frac{1}{\sin^2\xi}$

Magnitude in tensor notation:

$$A^2 = g_{ij} A^i A^j = g_{11}(A^1)^2 + g_{22}(A^2)^2 + g_{33}(A^3)^2$$

Dot product in tensor notation: $\vec{A}.\vec{B} = g_{ij} A^i B^j$

Angle between two vectors in tensor notation:

$$\cos\theta = \frac{g_{ij} A^i B^j}{\sqrt{g_{ij} A^i A^j}\sqrt{g_{ij} B^i B^j}}$$

GATE Previous Years Questions with Solutions

1. S_{ij} and A_{ij} represent a symmetric and anti-symmetric real valued tensor respectively in three dimension. The number of independent components of S_{ij} and A_{ij} are : **[GATE-2003]**

(a) 3 and 6 respectively (b) 6 and 3 respectively
(c) 6 and 6 respectively (d) 9 and 6 respectively

Soln. Number of independent components of a symmetric tensor of rank 2 in 3-D will be $\frac{1}{2}3(3+1)=6$.

Number of independent components of an anti-symmetric tensor of rank 2 in 3-D will be $\frac{1}{2}3(3-1)=3$.

Correct option is (b)

2. Let $T_{ij}=\sum_k \varepsilon_{ijk}a_k$ and $\beta_k=\sum_{i,j}\varepsilon_{ijk}T_{ij}$, where ε_{ijk} is the Leci Civia density, defined to be zero, if two of the indices coincide and +1 and –1 depending on whether *ijk* is even or odd permutation of 1, 2, 3. Then β_3 is equal to **[GATE-2009]**

(a) $2a_3$ (b) $-2a_3$ (c) a_3 (d) $-a_3$

Soln. $\beta_3=\sum_{i,j}\varepsilon_{ij3}T_{ij}=\varepsilon_{113}T_{11}+\varepsilon_{223}T_{22}+\varepsilon_{333}T_{33}+\varepsilon_{123}T_{12}+\varepsilon_{213}T_{21}+\varepsilon_{233}T_{23}+\varepsilon_{323}T_{32}+\varepsilon_{133}T_{13}+\varepsilon_{313}T_{31}$

$$=\varepsilon_{123}T_{12}+\varepsilon_{213}T_{21}=T_{12}-T_{21}$$

$$\Rightarrow \beta=\sum_k\varepsilon_{12k}a_k-\sum_k\varepsilon_{21k}a_k=\varepsilon_{123}a_3-\varepsilon_{213}a_3=2a_3$$

Correct option is (a)

3. Consider an anti-symmetric tensor P_{ij} with the indices *i* and *j* running from 1 to 5. The number of independent components of the tensor is **[GATE-2010]**

(a) 3 (b) 10 (c) 9 (d) 6

Soln. P_{ij} is an anti-symmetric tensor of rank 2 in dimension 5. Therefore, number of independent components

$$=\frac{1}{2}n(n-1)=\frac{1}{2}\times5\times4=10$$

Correct option is (b)

4. The number of independent components of the symmetric A_{ij} with indices i, j = 1, 2, 3 is

(a) 1 (b) 3 (c) 6 (d) 9 **[GATE-2012]**

Soln. The number of independent components of a symmetric tensor of rank 2 in 3-dimensional space will be

$$\frac{1}{2}3(3+1)=6$$

Correct option is (c)

5. In the most general case, which one of the following quantities is not a second order tensor ?

(a) Stress (b) Strain (c) Moment of inertia (d) Pressure **[GATE-2013]**

Soln. Stress = Force/Area, there are two directions involved with it, i.e., one is direction of force and the other one is direction of area.

Strain = change in length/original length, there are two directions involved with it, i.e., one is the direction of length and the direction along which length is changing.

Moment of inertia : can be defined as $I=L/w$ i.e., two directions involved one of angular momentum L and other of angular velocity w.

Pressure = Force/Area ; But here direction of force and direction of area are same. Therefore, it is a first order tensor.

Correct option is (d)

6. The length element *ds* of an arc is given by

$$(ds)^2=2(dx^1)^2+(dx^2)^2+\sqrt{3}(dx^1)(dx^2)$$

The metric tensor g_{ij} is **[GATE-2014]**

(a) $\begin{pmatrix}2 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$ (b) $\begin{pmatrix}2 & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & 1\end{pmatrix}$ (c) $\begin{pmatrix}2 & 1\\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$ (d) $\begin{pmatrix}1 & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & 2\end{pmatrix}$

Soln. $(ds)^2=2(dx^1)^2+(dx^2)^2+\sqrt{3}(dx^1)(dx^2)$

General form of metric tensor g_{ij} in 2-D will be $(ds)^2=g_{11}(dx^1)^2+g_{22}(dx^2)^2+2g_{12}(dx^1)(dx^2)$

(Here $g_{12}=g_{21}$)

So, metric tensor $g_{ij}=\begin{bmatrix}2 & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & 1\end{bmatrix}$

Correct option is (b)

PRACTICE SET

1. Which of the following is a tensor of order '0'?
   (a) $\vec{A}+\vec{B}$ (b) $\vec{A}-\vec{B}$ (c) $\vec{A}.\vec{B}$ (d) $\vec{A}\times\vec{B}$

2. Which of the following statements is/are true?
   (a) Gradient of scalar field is a contravariant tensor of rank 1
   (b) Gradient of scalar field is a covariant tensor of rank 1
   (c) Velocity is contravariant tensor of rank 1
   (d) Velocity is contravariant tensor of rank 2

3. If A is a scalar field, show that the second order derivatives of A i.e $\frac{\partial^2 A}{\partial x^i \partial x^j}$ are not the components of a tensor.

4. The number of components for a tensor of rank 3 is
   (a) 9 (b) 27 (c) 81 (d) 3

5. The Kroenker delta tensor δ_i^j can be expressed as
   (a) $\frac{\partial x^j}{\partial \bar{x}^k}\frac{\partial \bar{x}_k}{\partial x^i}$ (b) $\frac{\partial x^j}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^i}$ (c) $\frac{\partial x^j}{\partial \bar{x}_k}\frac{\partial \bar{x}^k}{\partial x^i}$ (d) $\frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^j}$

6. In contraction process,
   (a) the rank of the tensor is reduced by 2 .
   (b) the rank of the tensor is reduced by 1.
   (c) the rank of the tensor can be reduced by any number.
   (d) the rank of the tensor is increased by 2.

7. Which of the following is/are correct for Kroncker delta tensor δ_{ij} and Alternate tensor ε_{ijk} ?
   (a) $\varepsilon_{ijk}\delta_{ij}=1$ (b) $\delta_{ij}\varepsilon_{ijk}=1$ (c) $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$ (d) $\varepsilon_{ijl}\varepsilon_{kjl}=2\delta_{ik}$

8. Find the metric tensor for the following:
   (a) a two dimensional plane in cartesian co-ordinate system.
   (b) a two dimensional plane in plane polar co-ordinate system.
   (c) three dimensional cartesian co-ordinate system.
   (d) three dimensional spherical polar co-ordinate system.
   (e) three dimensional cyllindrical co-ordinate system.

ANSWER KEY

1. (c) **2. (a), (c)** **4. (b)** **5. (b)** **6. (a)** **7.(c),(d)**

Chapter 11

GROUP THEORY

Definition of Group:

A group is a collection of distinct numbers or of elements following a law of composition (addition or multiplication) and the element should follow the following properties:

(1) If A ∈ G, B ∈ G ⇒ AoB, BoA ∈ G ('o' is sign of law of composition)

(2) Their should be an element E (identity element) such that AoE = EoA = A and $E \in G$ Identity element of a group is unique.

(3) Their should be an element X (inverse element) corresponding to each element of the group, such that AoX = XoA = E

(4) If A, B, C ∈ G ⇒ (AoB)oC = Ao(BoC)

Example 1: Show that, G = {1, i, –1, –i} is a group under multiplication.

Soln: **Group Multiplication Table:**

	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

i.e. for any two elements a and b , $aob \in G$

Identity element of the group will be 1.

element	inverse element
1	1
i	$-i$
-1	-1
$-i$	i

Order of Group: It is defined as the number of elements present in the group.

Example 2: G = {1, ω, ω^2} of order 3 under multiplication.

Soln: Group Multiplication Table:

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Identity element of the group is 1

element	Inverse element
1	1
ω	ω^2
ω^2	ω

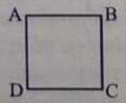
Abelian Group:

The group for which AoB = BoA for any two elements A and B of the group.

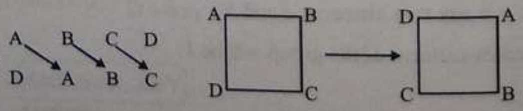
The groups given in example 1 and 2, are abelian in nature.

Group of symmetry elements of a square (C_{4v} group):

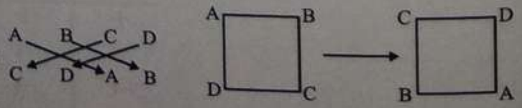
(a) Rotational symmetry elements: (Rotation about an axis passing through the center of the square and perpendicular to the plane of the paper)



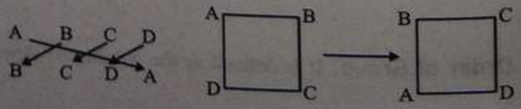
(i) C_4: Clockwise rotation about 90°:



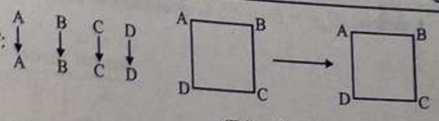
(ii) C_4^2 : Clockwise rotation about 180°:



(iii) C_4^3: Clockwise rotation about 270°:

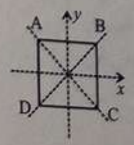


(iv) C_4^4 : Clockwise rotation about 360°:



(E identity element)

(b) Reflection Symmetry elements:



(i) m_x : Reflection on x-axis

(ii) m_y: Reflection on y-axis

(iii) σ_A : Reflection on AC diagonal

(iv) σ_B : Reflection on BD diagonal

	A	B	C	D
m_x:	D	C	B	A
m_y:	B	A	D	C
σ_A :	A	D	C	B
σ_B :	C	B	A	D

These eight elements form a group of order eight under multiplication.

$$G = \{E, C_4, C_4^2, C_4^3, m_x, m_y, \sigma_A, \sigma_B\}$$

Property 1: $C_4 \in G, C_4^2 \in G \Rightarrow C_4.C_4^2 = C_4^3 \in G$

$C_4 \in G, m_y \in G \Rightarrow C_4 m_y(A\ B\ C\ D) = C_4(B\ A\ D\ C) = (\mathrm{C\ B\ A\ D}) = \sigma_B(\mathrm{A\ B\ C\ D})$

$C_4 m_y(ABCD) = \sigma_B(ABCD) \Rightarrow C_4 m_y = \sigma_B \in G$

Property 2: $C_y E = C_y$, $\sigma_x E = \sigma_x$, here E is identity element.

Property 3:

Element	Inverse Element
C_4	C_4^3
m_x	m_x
σ_x	σ_x
C_4^2	C_4^2

Orders of the elements of the group:

If for an element A of the group, $A^n = E$, (where E is the identity element of the group) then 'n' is the order of element 'A' of the group.

Example: $(C_4)^4 = E \Rightarrow C_4$ is an element of order 4

element	order
E	1
C_4, C_4^3	4
C_4^2	2
m_x, m_y	2
σ_x, σ_y	

Conjugate Elements:

A, B, C $\in$ G, Such that $A^{-1}BA = C$, then B and C are conjugate elements of each other.

Example: Conjugate element of $C_4^3 = C_4$ as $m_x C_4^3 m_x = C_4$ and $m_y C_4^3 m_y = C_4$

Class of a Group:

We can split a group into sets such that all the elements of a set are conjugate to each other but no two elements belonging to different sets are conjugate each other. Such set of elements are known to be class of a group.

Example: $\{E\}, \{C_4, C_4^3\}, \{C_4^2\}, \{m_x, m_y\}, \{\sigma_x, \sigma_y\}$ are the classes of the group of symmetry elements of a square.

Example 3: If in a group G, the elements are satisfying the relation $x^2 = E$ (identity element). Show that, G is abelian in nature.

Soln: For any two elements x an y of the group G,

$$x^2 = E \Rightarrow x = x^{-1} \text{ and } y^2 = E \Rightarrow y = y^{-1}$$

$(xy) = (xy)^{-1} = y^{-1}\, x^{-1} \Rightarrow xy = yx$

So, the group G will be abelian in nature.

Cyclic Group:

A cyclic group is a group in which every element can be generated from a single element of the group and the single element is known as the generator of the cyclic group.

Example: i, –i are two genarators of the cyclic group $\{1, i, -1, -i\}$

$(i)^1 = i; (i)^2 = -1, (i)^3 = -i; (i)^4 = 1$

$(-i)^1 = -i; (-i)^2 = -1, (-i)^3 = i; (-i)^4 = 1$

All cyclic groups are Abelian in nature.

Example 4: Find the generators of the following cyclic group:

$$G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = E\}$$

Soln: a, a^3, a^5, a^7 are generators in this group G.

Example 5: Find the identity and inverse element of the element $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$ of a group under mulitplication. [where x = non-zero real number]

Soln: Let the identity element is $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$

So, $\begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \Rightarrow \begin{bmatrix} 2ax & 2ax \\ 2ax & 2ax \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \Rightarrow 2ax = x \Rightarrow a = \frac{1}{2}$

Identify element of the group will be $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Let the inverse Element is $\begin{bmatrix} b & b \\ b & b \end{bmatrix}$

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 2bx & 2bx \\ 2bx & 2bx \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow 2bx = \frac{1}{2} \Rightarrow b = \frac{1}{4x}$$

Inverse element will be = $\begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$

Example 6: If G is Abelian and $a, b \in G$ such that $O(a) = 5$, $O(b) = 7$, then $(ab)^{14} = ??$

Soln: Given, $ab = ba$ and $a^5 = E$, $b^7 = E$

$(ab)^{14} = a^{14}\, b^{14} = (a^5)^2.a^4.(b^7)^2 = E^2.a^4.E^2 = a^4 = a^{-1}$

Sub-Group:

A set 'H' is said to be subgroup of G if H itself is a group under the same law of composition as G and all elements in H will be also is G.

Example: G = $\{E, C_4, C_4^2, C_4^3, m_x, m_y, \sigma_A, \sigma_B\}$

Subgroups of G: $\{E, C_4^2, m_x, m_y\}, \{E_1, C_4^2\}, \{E_1, m_x\}, \{E, m_y\}, \{E, \sigma_x\}, \{E, \sigma_y\}$

Identity element of the sub-group H is same as the main group G.

Identity element and total group will always form a sub-group of the main group G.

Proper Sub-group:

H is proper sub-group of G if H ≠ G i.e. number of elements in H ≠ number of elements in G

Improper Sub-group:

H is improper sub-group of G if H = G i.e. Number of elements in H = number of elements in G

Isomorphic Group:

If there are two groups $G = \{E, A, B, C\}$, $G' = \{E', A', B', C'\}$ of same order and there is a one to one correspondence between the elements of G and G' i.e.

$$E \to E', A \to A', B \to B', C \to C'$$

then G and G' are said to be isomorphic to each other. It is Similar to one to one mapping.

Example: $\mathrm{G} = \{\mathrm{E}, \mathrm{C}_4, \mathrm{C}_4^2, \mathrm{C}_4^3\}$ & $\mathrm{G}' = \{1, \mathrm{i}, -1, -\mathrm{i}\}$

The group multiplication table of G and G' are as follows:

	E	C_4	C_4^2	C_4^3
E	E	C_4	C_4^2	C_4^3
C_4	C_4	C_4^2	C_4^3	E
C_4^2	C_4^2	C_4^3	E	C_4
C_4^3	C_4^3	E	C_4	C_4^2

	1	i	-1	$-i$
1	1	i	-1	$-i$
$-i$	$-i$	1	i	-1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

From the group multiplication table we can say $\mathrm{E} \to 1, \mathrm{C}_4 \to \mathrm{i}, \mathrm{C}_4^2 \to -1, \mathrm{C}_4^3 \to -\mathrm{i}$

Homomorphic Group:

If there are two groups $\mathrm{G} = \{\mathrm{E}, \mathrm{A}, \mathrm{B}, \mathrm{C}\}$, $G' = \{E', A'\}$ such that several elements of G corresponds to a single element of G' i.e.

$$\mathrm{E}, \mathrm{A} \to \mathrm{E}', \mathrm{B}, \mathrm{C} \to \mathrm{A}'$$

then G and G' are said to be homomorphic to each other. It is Similar to many to one mapping.

Example: $\mathrm{G} = \{\mathrm{E}, \mathrm{C}_4^2, \mathrm{m}_x, \mathrm{m}_y\}$ & $\mathrm{G}' = \{\mathrm{E}, \mathrm{m}_x\}$

The group multiplication of G and G' are as folllows:

	E	C_4^2	m_x	m_y
E	E	C_4^2	m_x	m_y
C_4^2	C_4^2	E	m_y	m_x
m_x	m_x	m_y	E	C_4^2
m_y	m_y	m_x	C_4^2	E

	E	m_x
E	E	m_x
m_x	m_x	E

From the group multiplication table, we can say

$$E, C_4^2 \to E \quad \text{and} \quad m_x, m_y \to m_x$$

Full Linear Group (FLG):

The collection of all non-singular matrices of order n, with matrix multiplication as law of composition is called full linear group and the order of the group is generally infinite.

$U(n)$ group:

Sub-group of full linear group is unitary group whose elements are unitary matrices of order n.

$SU(n)$ group:

Special unitary group of unitary matrices of order 'n' and determinant equal to 1

$SU(2)$ group:

Special unitary group of unitary matrices of order 2 and determinant equal to 1

Suppose $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \Rightarrow UU^\dagger = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} aa^*+bb^* & ac^*+bd^* \\ a^*c+b^*d & cc^*+dd^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, $aa^*+bb^*=1$ Since, $ad-bc=1$

$ac^*-bd^*=0$

Therefore, $d=a^*, -c=b^* \Rightarrow c=-b^*$

So, the general form of the elements of a $SU(2)$ group is $U = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$

$O(n)$ Group:

Group of orthogonal matrices of order 'n'

$O(3)$ Group:

Group of orthogonal matrices of order 3.

CSIR-NET/JRF Previous Years Questions with Solutions

1. Which of the following matrices is an element of the group SU(2)? **[CSIR JUNE 2011]**

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1+\mathrm{i}}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-\mathrm{i}}{\sqrt{3}} \end{pmatrix}$ (c) $\begin{pmatrix} 2+\mathrm{i} & \mathrm{i} \\ 3 & 1+\mathrm{i} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Soln. A matrix $U(a,b)=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, with det (U) = 1 is called an element of the special unitary 2×2 group SU(2).

The matrix $\begin{pmatrix} \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$ belongs to SU(2).

Determinant of the matrix $= aa^* + bb^*$

$$= |a|^2 + |b|^2 = \frac{|1+i|^3}{3} + \frac{1}{3} = \frac{1+1}{3} + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1$$

Correct option is (b)

2. Let α and β be complex numbers. Which of the following sets of matrices forms a group under matrix multiplication?

(a) $\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$, where $\alpha\beta \neq 1$

(c) $\begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}$, where $\alpha\beta^*$ is real (d) $\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$, where $|\alpha|^2 + |\beta|^2 = 1$

Soln. SU(2) is a group of special unitary matrices (having determinant 1) of order 2, under matrix multiplication as the law of composition. The elements of this group can be represented as

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

where the determinant of the matrix $= \alpha\alpha^* + \beta\beta^* = |\alpha|^2 + |\beta|^2 = 1$

Correct Option is (d)

PRACTICE SET

Consider the following set $\{1,-1,i,-i\}$

1. Which of the following statements is/are CORRECT?
(a) The set is a finite non-abelian group under multiplication.
(b) The set is a finite abelian group under multiplication.
(c) The set is a finite abelian group under addition.
(d) The set is a finite non-abelian group under addition.

2. If a group is defined by $a*b=a+b-1$, then the inverse of the group is
(a) $2-a$ (b) a (c) a^{-1} (d) 0

3. In a group G, the elements are satisfying the relation $x^2 = E$ (identity element). Show that G is abelian in nature.

4. If G is an abelian group and $a,b \in G$ such that $O(a)=5$ and $O(b)=7$, then $(ab)^{14}$ is
(a) a (b) a^{-1} (c) a^2 (d) a^3

5. The generators of the following group:

$$G=\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8=E\}$$

(a) a,a^2,a^4 (b) a^2,a^4,a^6 (c) a,a^3,a^5,a^7 (d) a^2,a^4,a^6,a^8

6. (i) Show that, the set of all nth roots of unity forms a finite abelian group under multiplication.
(ii) Check that if it is cyclic or not. If yes, find its one generator.

7. The special unitary group of matrices i.e. $SU(2)$ can be represented by

(a) $\begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ (c) $\begin{bmatrix} a & b \\ -b^* & -a^* \end{bmatrix}$ (d) $\begin{bmatrix} a & b \\ b^* & -a^* \end{bmatrix}$

8. Which of the following matrices is an element of the group SU(2)?

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{bmatrix}$ (c) $\begin{bmatrix} 2+i & i \\ 3 & 1+i \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

ANSWER KEY

1. (b) 2. (a) 4. (b) 5. (c) 7. (b) 8 .(b)

Chapter 12

DIRAC DELTA , BETA & GAMMA FUNCTION

Dirac Delta function:

$$\delta(x-a)=0 \quad x\neq a$$
$$=\infty \quad x=a \quad (\text{very high})$$

Such that $\int_{-\infty}^{\infty}\delta(x-a)\,dx=1$ i.e. area under the curve should be equal to unity.

Some representations of Dirac Delta function:

(1) $\lim_{\sigma\to 0}R_\sigma(x)=\delta(x-a)$ where $R_\sigma(x)=\frac{1}{2\sigma} \quad a-\sigma<x<a+\sigma$
$=0 \quad$ otherwise

(2) $\lim_{\sigma\to 0}G_\sigma(x)=\delta(x-\mu)$ where $G_\sigma(x)=\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$

Some properties of Dirac Delta function:

(1) $\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$

(2) $\int_{-\infty}^{\infty}f'(x)\delta(x-a)dx=-f'(a)$

(3) $\delta(x-a)=\delta(x-a)$

(4) $\delta[c(x-a)]=\frac{1}{|c|}\delta(x-a) \quad$ (c is constant)

(5) $\delta(x^2-a^2)=\frac{1}{|2a|}[\delta(x-a)+\delta(x+a)]$

(6) $\int_{-\infty}^{\infty}\delta(x-a)\delta(x-b)dx=\delta(a-b)$

(7) $F[\delta(x-a)]=\frac{1}{\sqrt{2\pi}}e^{isa}$

(8) $L[\delta(x-a)]=e^{-sa}$

Example 1: Evaluate the following integrals

Soln. (i) $\int_0^{\pi}\cos x\,\delta(x-\pi)\,dx=\cos\pi=-1$

(ii) $\int_{-2}^{2}x^3\delta(x+1)\,dx=(-1)^3=-1$

(iii) $\int_{-2}^{2}(2x+3)\delta(3x)\,dx=\frac{1}{3}\int_{-2}^{2}(2x+3)\delta(x)\,dx=1$

(iv) $\int_0^{2}(x^3+3x+2)\delta(x-1)\,dx=2$

(v) $\int_{-\infty}^{\infty}\cos x\,\delta(x^2-\pi^2)\,dx=\int_{-\infty}^{\infty}\cos x\frac{1}{|2\pi|}[\delta(x-\pi)+\delta(x+\pi)]dx=\frac{1}{2\pi}[\cos\pi+\cos(-\pi)]=-\frac{1}{\pi}$

Gamma Function: $\int_0^{\infty}e^{-x}x^{n-1}dx=\Gamma n$

Some important relations:

(1) $\Gamma(n+1)=n\Gamma n=n!$ (2) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

(3) Gamma of zero and negative integers is not defined.

Example 2: Evaluate the following integral $\int_0^{\infty}x^2e^{-2x^2}dx$

Soln. $\int_0^{\infty}x^2e^{-2x^2}dx$, assuming $2x^2=t\Rightarrow 4xdx=dt \quad \Rightarrow dx=\frac{dt}{4x}=\frac{1}{2\sqrt{2t}}dt$, we get

$$\frac{1}{4\sqrt{2}}\int_0^{\infty}e^{-t}t^{1/2}dt=\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)=\frac{1}{8\sqrt{2}}\sqrt{\pi}$$

Beta function:

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx=\beta(m,n)$$

Some important relations:

(1) $\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ (2) $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

(where p and q are positive integers of fractions)

(3) $\Gamma x.\Gamma(1-x)=\frac{\pi}{\sin\pi x}$

Example 3: Evaluate the integral $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

Soln. $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta=\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}=\frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{\sqrt{2}}$

GATE Previous Years Questions and Solutions

1. The value of $\int_0^3 t^2\delta(3t-6)\,dt$ is_______ (upto one decimal place) **[GATE 2015]**

Soln. $\int_0^3 t^2\delta(3t-6)\,dt \int_0^3 t^2\delta\left[3(t-2)\right]dt = \frac{1}{3}\int_0^3 t^2\delta(t-2)\,dt = \frac{4}{3} = 1.3$

2. $\Gamma\left(n+\frac{1}{2}\right)$ is equal to [Given : $\Gamma(n+1) = n\,\Gamma(n)$ and $\Gamma(1/2) = \sqrt{\pi}$ **[GATE-2013]**

(a) $\frac{n!}{2}\sqrt{\pi}$ (b) $\frac{2n!}{n!2^n}\sqrt{\pi}$ (c) $\frac{2n!}{n!2^{2n}}\sqrt{\pi}$ (d) $\frac{n!}{2^{2n}}\sqrt{\pi}$

Soln. For $n=0$, $\sqrt{\frac{1}{2}} = \sqrt{\pi}$

For $n=1$, $\sqrt{\frac{3}{2}} = \frac{1}{2}\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{\pi}$

For $n=2$, $\sqrt{\frac{5}{2}} = \frac{3}{2}\sqrt{\frac{3}{2}} = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

All these satisfies the form given in option (c)

Correct option is (c)

Chapter 13

PARTIAL DIFFERENTIATION

Introduction:

Let z denote the function f(x,y) which posseses one definite value for every pair of values of x and y, and moreover if y is kept constant, $z = f(x,y)$ varies because of x, and similarly if x is kept constant, z varies because of the variation in a single variable y. A similar definition can be given for functions of more than two variables, i.e., $u = f(x,y,z)$ denotes a function of three variables x, y, z.

For functions of two variable, the region of definition is a portion of a plane, the (x,y) plane.

For functions of three variables, the region of definition is a portion of a three dimensional space and so on. $Ax + By + Cz = D$ is a function of three variables. It represents a plane, $x^2 + y^2 + z^2 = r^2$ represents a sphere

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ represents the solid ellipsoid.

Partial derivatives of *f* (x,y):

The ordinary derivative of $f(x,y)$ with respect to x keeping the other variable constant is called the partial derivtive of $f(x,y)$ with respect to x. Similarly ordinary derivative of $f(x,y)$ with respect to y keeping the other variable x constant is called the partial derivative of $f(x,y)$ with respect to y.

The partial derivative of $f(x,y)$ with respect to x and y are generally denoted by $f_x = \frac{\partial f}{\partial x}$ and

$f_y = \frac{\partial f}{\partial y}$ .Thus by definition we write

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(x,k+y)-f(x,y)}{k}$$

when these limits exist.

At a particular point (a,b), these partial derivatives are often denoted by $f_x(a,b)$ and $f_y(a,b)$. Thus,

$$f_x(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h};\ f_y(a,b) = \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}$$

Again $f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h};\ f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$

provided the limit exist.

Example 1: Find the first and second order partial derivatives of x^3+y^3+3axy

Soln: Let $f(x,y)=x^3+y^3+3axy$, then $\frac{\partial f}{\partial x}=3x^2+3ay,\ \frac{\partial f}{\partial y}=3y^2+3ax$

$$\frac{\partial^2 f}{\partial y\partial x}=3a,\qquad \frac{\partial^2 f}{\partial x\partial y}=3a;\ \frac{\partial^2 f}{\partial x^2}=6x,\qquad \frac{\partial^2 f}{\partial y^2}=6y;$$

Example 2: If $u=e^{xyz}$, show that $\frac{\partial^3 u}{\partial x\partial y\partial z}=\left(1+3xyz+x^2y^2z^2\right)e^{xyz}$

Soln: We have $u=e^{xyz}$, $\Rightarrow \frac{\partial u}{\partial x}=e^{xyz}.yz \Rightarrow \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(yze^{xyz}\right)=z\left[ye^{xyz}.xz+e^{xyz}\right]$

$$\Rightarrow \frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x\partial y}\right)=\frac{\partial}{\partial z}\left[xyz^2e^{xyz}+ze^{xyz}\right]=xy\left[z^2e^{xyz}.xy+e^{xyz}.2z\right]+\left[ze^{xyz}.xy+e^{xyz}\right]$$

$$=e^{xyz}\left(1+3xyz+x^2y^2z^2\right)$$

Example 3: If $u=\log\left(x^3+y^3+z^3-3xyz\right)$ show that

(a) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=-\frac{9}{(x+y+z)^2}$ and (b) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{z}{(x+y+z)^2}$

Soln. (a) $u=\log\left(x^3+y^3+z^3-3xyz\right)\Rightarrow \frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$

$$\Rightarrow \frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}\quad\text{and}\quad \frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}=\frac{3}{x+y+z}$$

Now, $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}=-\frac{9}{(x+y+z)^2}$$

(b) $\frac{\partial^2 u}{\partial x^2}=\frac{6x\left(x^3+y^3+z^3-3xyz\right)-\left(3x^2-3yz\right)\left(3x^2-3yz\right)}{\left(x^3+y^3+z^3-3xyz\right)^2}=-\frac{3\left(x^4-2xy^3-2xz^3+3y^2z^2\right)}{\left(x^3+y^3+z^3-3xyz\right)^2}$

Similarly, $\frac{\partial^2 u}{\partial y^2}=-\frac{3\left(y^4-2yz^3-2yx^3+3z^2x^2\right)}{\left(x^3+y^3+z^3-3xyz\right)^2}$ and $\frac{\partial^2 u}{\partial z^2}=-\frac{3\left(z^4-2zx^3-2zy^3+3x^2y^2\right)}{\left(x^3+y^3+z^3-3xyz\right)^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)^2}{\left(x^3+y^3+z^3-3xyz\right)^2}=-\frac{3}{(x+y+z)^2}$$

Example 4: If $u=\left(x^2+y^2+z^2\right)^{-1/2}$, prove that

(a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u$; (b) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

Soln: $u=\frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}\Rightarrow \frac{\partial u}{\partial x}=-\frac{1}{2\left(x^2+y^2+z^2\right)^{3/2}}.2x=-\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}$

$$\Rightarrow x\frac{\partial u}{\partial x}=-\frac{x^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Similarly, $y\frac{\partial u}{\partial y}=-\frac{y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$ and $z\frac{\partial u}{\partial z}=-\frac{z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-\frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}=-\frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}=-u$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left\{-\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}=-\left\{\frac{1.\left(x^2+y^2+z^2\right)^{3/2}-x.\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}.2x}{\left(x^2+y^2+z^2\right)^3}\right\}$$

$$=-\frac{\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^3}\left(x^2+y^2+z^2-3x^2\right)=-\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(y^2+z^2-2x^2\right)$$

Similarly, $\frac{\partial^2 u}{\partial y^2}=-\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(z^2+x^2-2y^2\right)$ and $\frac{\partial^2 u}{\partial z^2}=-\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(x^2+y^2-2z^2\right)$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2\right)=0$$

Example 5: If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, prove that $u_x^2+u_y^2=2\left(xu_x+yu_y+zu_z\right)$

Soln: Differentiating both sides of the given relation with respect to x partially, we get

$$\frac{2x}{a^2+u}-\left\{\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}\right\}\frac{\partial u}{\partial x}=0$$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{2x/\left(a^2+u\right)}{\left\{\frac{x^2}{\left(a^2+u\right)^2}+\frac{y^2}{\left(b^2+u\right)^2}+\frac{z^2}{\left(c^2+u\right)^2}\right\}}$$

$$\Rightarrow u_x^2=\left(\frac{\partial u}{\partial x}\right)^2=\frac{\dfrac{4x^2}{\left(a^2+u\right)^2}}{\left\{\dfrac{x^2}{\left(a^2+u\right)^2}+\dfrac{y^2}{\left(b^2+u\right)^2}+\dfrac{z^2}{\left(c^2+u\right)^2}\right]^2}$$

$$\Rightarrow \mathrm{u}_x^2+\mathrm{u}_y^2+\mathrm{u}_z^2=\frac{4\left[\dfrac{x^2}{\left(a^2+u\right)^2}+\dfrac{y^2}{\left(b^2+u\right)^2}+\dfrac{z^2}{\left(c^2+u\right)^2}\right]}{\left[\dfrac{x^2}{\left(a^2+u\right)^2}+\dfrac{y^2}{\left(b^2+u\right)^2}+\dfrac{z^2}{\left(c^2+u\right)^2}\right]^2}=\frac{4}{\left\{\dfrac{x^2}{\left(a^2+u\right)^2}+\dfrac{y^2}{\left(b^2+u\right)^2}+\dfrac{z^2}{\left(c^2+u\right)^2}\right\}}$$

Now, $2x\dfrac{\partial u}{\partial x}=\dfrac{\dfrac{4x^2}{a^2+u}}{\left\{\dfrac{x^2}{\left(a^2+u\right)^2}+\dfrac{y^2}{\left(b^2+u\right)^2}+\dfrac{z^2}{\left(c^2+u\right)^2}\right\}}$

$$\Rightarrow 2\left(xu_x+yu_y+zu_z\right)=\frac{4\left[\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}\right]}{\left\{\dfrac{x^2}{\left(a^2+u\right)^2}+\dfrac{y^2}{\left(b^2+u\right)^2}+\dfrac{z^2}{\left(c^2+u\right)^2}\right\}}$$

$$=\frac{4}{\left\{\dfrac{x^2}{\left(a^2+u\right)^2}+\dfrac{y^2}{\left(b^2+u\right)^2}+\dfrac{z^2}{\left(c^2+u\right)^2}\right\}}\quad\left(\because \frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1\right)$$

Therefore, $\left(u_x^2+u_y^2+u_z^2\right)=2\left(xu_x+yu_y+zu_z\right)$

Example 6: If $x^x.y^y.z^z=c$ show that at $x=y=z$, $\dfrac{\partial^2 z}{\partial x\partial y}=-\left(x\log ex\right)^{-1}$

Soln: Given that, $x^xy^yz^z=c$

Taking log of both sides, we have, $x\log x+y\log y+z\log z=\log c$

Differentiating partially w.r.t. x, we get,

$$x.\frac{1}{x}+\log x+\left(z.\frac{1}{z}+\log z\right)\frac{\partial z}{\partial x}=0 \quad\Rightarrow\quad \frac{\partial z}{\partial x}=-\frac{\left(1+\log x\right)}{\left(1+\log z\right)}$$

Similarly, we have, $\dfrac{\partial z}{\partial y}=-\dfrac{\left(1+\log y\right)}{\left(1+\log z\right)}$

$$\Rightarrow \frac{\partial^2 z}{\partial x\partial y}=\frac{\left(1+\log x\right)}{\left(1+\log z\right)^2}.\frac{1}{z}\frac{\partial z}{\partial y}=-\frac{\left(1+\log x\right)}{\left(1+\log z\right)^2}.\frac{1}{z}.\frac{\left(1+\log y\right)}{\left(1+\log z\right)}=-\frac{\left(1+\log x\right)^2}{\left(1+\log x\right)^3}.\frac{1}{x}\ (\text{since } x=y=z)$$

$$=-\frac{1}{x\left(1+\log x\right)}=-\frac{1}{x\log ex}=-\left(x\log ex\right)^{-1}$$

Example 7: If $z=x^2\tan^{-1}\left(y/x\right)-y^2\tan^{-1}\left(x/y\right)$, prove that $\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{x^2-y^2}{x^2+y^2}$

Soln: We have, $\dfrac{\partial z}{\partial x}=2x\tan^{-1}\dfrac{y}{x}+\dfrac{x^2}{1+\left(y/x\right)^2}\left(-\dfrac{y}{x^2}\right)-\dfrac{y^2}{1+\left(x/y\right)^2}\left(\dfrac{1}{y}\right)$

$$=2x\tan^{-1}\left(y/x\right)-\frac{x^2y}{x^2+y^2}-\frac{y^3}{x^2+y^2}=2x\tan^{-1}\left(y/x\right)-\frac{y\left(x^2+y^2\right)}{\left(x^2+y^2\right)}=2x\tan^{-1}\left(y/x\right)-y$$

Now, $\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial y}\left\{2x\tan^{-1}\dfrac{y}{x}-y\right\}=\dfrac{2x}{1+\left(y/x\right)^2}\left(\dfrac{1}{x}\right)-1=\dfrac{2x^2}{x^2+y^2}-1=\dfrac{x^2-y^2}{x^2+y^2}$

Example 8: If $u=f\left(r\right)$, where $r^2=x^2+y^2$, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f'\left(r\right)+\frac{1}{r}f'\left(r\right)$$

Soln: We have, $2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$ and $2\mathrm{r}\dfrac{\partial r}{\partial y}=2y$ or $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$

Now, $u=f\left(r\right)\Rightarrow\dfrac{\partial u}{\partial x}=f'\left(r\right).\dfrac{\partial r}{\partial x}=f'\left(r\right)\cdot\dfrac{x}{r}$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{f'\left(r\right)x}{r}\right]=\frac{r\left\{f'\left(r\right).1+x\,f''\left(r\right)\left(\dfrac{\partial r}{\partial x}\right)\right\}-x\,f'\left(\dfrac{\partial r}{\partial x}\right)}{r}=\frac{1}{r^2}\left[r\,f'\left(r\right)+x^2f''\left(r\right)-\frac{x^2}{r}f'\left(r\right)\right]$$

Similary, $\dfrac{\partial^2 u}{\partial y^2}=\dfrac{1}{r^2}\left[r\,f'\left(r\right)+y^2f''\left(r\right)-\dfrac{y^2}{r}f'\left(r\right)\right]$

Hence, $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial u}{\partial y^2}=\dfrac{1}{r^2}\left[2rf'\left(r\right)+\left(x^2+y^2\right)f''\left(r\right)-\dfrac{\left(x^2+y^2\right)}{r}f'\left(r\right)\right]$

$$=\frac{1}{r^2}\left[2rf'\left(r\right)+r^2f''\left(r\right)-rf'\left(r\right)\right]=\frac{1}{r}f'\left(r\right)+f''\left(r\right)$$

Example 9: If $\theta=t^ne^{-r^2/4t}$, find what value of n makes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2.\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$$

Soln: We have, $\theta = t^n e^{-r^2/4t} \Rightarrow \frac{\partial \theta}{\partial r} = t^n . e^{-r^2/4t}\left(-\frac{2r}{4t}\right) = -\frac{1}{2} r t^{n-1} e^{-r^2/4t}$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left[-\frac{1}{2} r^3 t^{n-1} e^{-r^2/4t}\right] = -\frac{1}{2} t^{n-1}\left[r^3 . e^{-r^2/4t}\left(-\frac{2r}{4t}\right) + 3r^2 . e^{-r^2/4t}\right]$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{4} t^{n-2}\left[r^2 - 6t\right] e^{-r^2/4t}$$

$$\frac{\partial \theta}{\partial t} = t^n . e^{-r^2/4t} \frac{r^2}{4t^2} + e^{-r^2/4t} n t^{n-1} = \frac{1}{4} t^{n-2} e^{-r^2/4t}\left(r^2 + 4nt\right)$$

Now to satisfy the condition given in the question, we have

$$\frac{1}{4} t^{n-2}\left[r^2 - 6t\right] e^{-r^2/4t} = \frac{1}{4} t^{n-2}\left(r^2 + 4nt\right) \Rightarrow r^2 - 6t = r^2 + 4nt \Rightarrow \mathrm{n} = -3/2$$

Homogeneous Functions:

An expression of the type $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{m-3} y^3 + ...a_n y^n$ in which every term is of degree n, is called a homogenous function of degree n in x and y.

Euler's theorem : If u is a homogenous function of degree 'n' in x and y then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

For any no. of variables, $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + ... + w \frac{\partial u}{\partial w} = nu$

Example 10: Verify Euler's theorem when $f(x, y, z) = 3x^2 yz + 5xy^2 z + 4z^4$

Soln: We have, $\frac{\partial f}{\partial x} = 6xyz + 5y^2 z$, $\frac{\partial f}{\partial y} = 3x^2 + 10xyz$, and $\frac{\partial f}{\partial z} = 3x^2 y + 5xy^2 + 16z^3$

Hence, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{df}{\partial z} = 6x^2 yz + 5xy^2 z + 3x^2 yz + 10xy^2 z + 3x^2 yz + 5xy^2 z + 16z^4$

$$= 4\left(3x^2 yz + 5xy^2 z + 4z^4\right) = 4f(x, y, z)$$

which verifies Euler's theorem.

Example 11: If u be a homogenous function of degree n, then show that

(i) $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$ (ii) $x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$

(iii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Soln: Since u is a homogeneous function of degree n, hence by Euler's theorem, we have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \quad ...(\mathrm{i})$$

(i) Differentiating both sides partialy with respect to x, we get

$$x \frac{\partial^2 u}{\partial x^2} + 1.\frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x} \Rightarrow x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x} \quad ...(\mathrm{ii})$$

(ii) Again differentiating (i) partially w.r.t. y

$$x \frac{\partial u}{\partial y \partial x} + y \frac{\partial u}{\partial y^2} + 1.\frac{\partial u}{\partial y} = n \frac{\partial u}{\partial y} \Rightarrow x \frac{\partial^2 u}{\partial x\, \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \quad ...(\mathrm{iii})$$

(iii) Mutiplying (ii) by x and (iii) by y and adding,

$$x \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) = n(n-1)u,$$

Example 12: If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

Soln: We have $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, *i.e*, $\tan u = \frac{x^3 + y^3}{x - y}$

Let $z = \tan u = \frac{x^3 + y^3}{x - y}$

Hence z is a homogeneous function of degree (3-1)=2; $\Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$

$$\Rightarrow x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 2\tan u \Rightarrow x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\sin u \cos u = \sin 2u$$

Example 13: If $V = \log_e \sin\left\{\frac{\pi\left(2x^2 + y^2 + xz\right)^{1/2}}{2\left(x^2 + xy + 2yz + z^2\right)^{1/3}}\right\}$. Find the value of

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} \text{ when } x = 0, y = 1, z = 2$$

Soln: Let $u = \frac{\pi\left(2x^2 + y^2 + xz\right)^{1/2}}{2\left(x^2 + xy + 2yz + z^2\right)^{1/3}} \Rightarrow \frac{\partial V}{\partial x} = \cot u \frac{\partial u}{\partial x}; \frac{\partial V}{\partial y} = \cot u \frac{\partial u}{\partial y}$ and $\frac{\partial V}{\partial z} = \cot u \frac{\partial u}{\partial z}$

$$\Rightarrow x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial x} = \cot u\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}\right) \quad ...(\mathrm{i})$$

But u is a homogenous function of degree 1/3 $\Rightarrow \mathrm{x} \frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y} \frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{z} \frac{\partial \mathrm{u}}{\partial \mathrm{z}} = \frac{1}{3}\mathrm{u}$

$$\Rightarrow \mathrm{x} \frac{\partial \mathrm{V}}{\partial \mathrm{x}} + \mathrm{y} \frac{\partial \mathrm{V}}{\partial \mathrm{y}} + \mathrm{z} \frac{\partial \mathrm{V}}{\partial \mathrm{z}} = \frac{1}{3}\mathrm{u} \cot \mathrm{u}.$$

where $x = 0, y = 1, z = 2$, we have $\mathrm{u} = \frac{1}{4}\pi$, and then, $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = \frac{1}{3}.\frac{\pi}{4}\cot\frac{\pi}{4} = \frac{\pi}{12}$

Example 14: If $u = \log\left\{\frac{\left(x^4 + y^4\right)}{x + y}\right\}$, show by Euler's theorem that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$

Soln: Suppose, $e^u = \frac{\left(x^4 + y^4\right)}{(x + y)} = z$

Now z is a homogenous function of degree (4-1)=3. Hence by Euler's theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial z}{\partial y}=3z \Rightarrow x\frac{\partial}{\partial x}\left(e^{u}\right)+y\frac{\partial}{\partial y}\left(e^{u}\right)=3e^{u} \Rightarrow e^{u}.x\frac{\partial u}{\partial x}+e^{u}.y\frac{\partial u}{\partial y}=3e^{u}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$$

Total differential coefficients or total derivatives :

Let $x=\phi(t)$, $y=\psi$ (t) define two functions for $t\left[t_0 \le t \le t_1\right]$. The pair (x, y) then define a corresponding region R in the (x.y) plane. Let u be the function of x and y such that $u=f(x,y)$ and $x=\phi(t)$ and $y=\psi(t)$.

Now u may vary because of variations in t and $\frac{du}{dt}$ at a point, if it exists, is called the total derivatives of u with respect to t, let δy and δu be the corresponding changes, for a change δ t in time.

We have, then

$$\frac{dx}{dt}=\lim_{\delta t\to 0}\frac{\delta x}{\delta t},\quad \frac{dy}{dt}=\lim_{\delta t\to 0}\frac{\delta y}{\delta t}\quad \text{and}\quad \frac{du}{dt}=\lim_{\delta t\to 0}\frac{\delta u}{\delta t}$$

Also as $\delta t\to 0, \delta x\to 0, \delta y\to 0$

Now, $\delta u=f(x+\delta x, y+\delta y)-f(x,y)$

Therefore, $$\frac{du}{dt}=\lim_{\delta t\to 0}\frac{\delta u}{\delta t}\text{s}$$

$$=\lim_{\substack{\delta x\to 0\\ \delta y\to 0\\ \delta t\to 0}}\frac{\left[f(x+\delta x, y+\delta y)-f(x,y+\delta y)+f(x,y+\delta y)-f(x,y)\right]}{\delta t}$$

$$=\lim_{\substack{\delta x\to 0\\ \delta y\to 0\\ \delta t\to 0}}\left[\frac{f(x+\delta x,y+\delta y)-f(x,y+\delta y)}{\delta x}.\frac{\delta x}{\delta t}+\frac{f(x,y+\delta y)-f(x,y)}{\delta y}.\frac{\delta y}{\delta t}\right]$$

$$\therefore \quad \frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{du}{dt} \qquad \text{... (i)}$$

Similarly if u is a function of x, y, z which are functions of a single variable t; then

$$\frac{du}{dt}=\frac{\delta u}{\delta x}.\frac{dx}{dt}+\frac{\delta u}{\delta y}.\frac{dy}{dt}+\frac{\delta u}{\delta z}.\frac{dz}{dt} \qquad \text{... (ii)}$$

Now suppose $t=x$, i.e., u is a function of x and y, where y is itself a function of x, then (i) becomes.

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx} \qquad \text{... (iii)}$$

If u is level curve, such that $u=f(x,y)=$ constant c, then $\left(\frac{du}{dx}=0\right)$

Therefore, (iii) becomes, $0=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=-\frac{\partial u/\partial x}{\partial u/\partial y}$

Similarly if $u=f(x_1,x_2,x_3,.....x_n)$ and $x_1,x_2,x_3,........x_n$ are all functions of t, we can prove that

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}.\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}.\frac{dx_2}{dt}+\frac{\partial u}{\partial x^3}.\frac{dx_3}{dt}+....+\frac{\partial u}{\partial x_n}.\frac{dx_n}{dt} \qquad \text{... (v)}$$

and also when u is a function of n variables $x_1,x_2,x_3.......x_n$ and $x_2,x_3,........x_n$ are all functions of x_1, then total differential coefficient.

$$\frac{du}{dx_1}=\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}.\frac{dx_2}{dx_1}+...+\frac{\partial u}{\partial x_n}.\frac{dx_n}{dx_1} \qquad \text{... (vi)}$$

To find $\frac{d^2y}{dx^2}$ from $f(x,y)=c$:

We know that $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$ i.e. $\frac{dy}{dx}=-\frac{f_x}{f_y}$ if $f_y \ne 0$

Now, $$\frac{d^2y}{dx^2}=-\frac{\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y\partial x}.\frac{dy}{dx}\right)\frac{\partial f}{\partial y}-\frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}.\frac{dy}{dx}\right)}{\left(\frac{\partial f}{\partial y}\right)^2}$$

$$=\frac{-\frac{\partial f}{\partial y}\left[\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y\partial x}.\left(-\frac{\partial f/\partial x}{\partial f/\partial y}\right)\right]+\frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f/\partial y^2\,\partial f/\partial x}{\partial f/\partial y}\right)}{\left(\frac{\partial f}{\partial y}\right)^2}$$

$$=\frac{-\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2+2\frac{\partial^2 f}{\partial y\partial x}.\frac{\partial f}{\partial y}.\frac{\partial f}{\partial x}-\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3}=\frac{-\left[f_x^2(f_y)^2-2f_{yx}f_xf_y+f_y^2(f_x)^2\right]}{(f_y)^3}$$

Also if p, q, r, s and t represent $f_x, f_y, f_x^2, f_{xy}, f_y^2$ then, $\frac{d^2y}{dx^2}=-\frac{\left(q^2r-2pqs+p^2t\right)}{q^3}$

Example 15: Find $\frac{dz}{dt}$ when $z=xy^2+x^2y, x=at^2, y=2at$

Soln: We have, $z=xy^2+x^2y \Rightarrow \frac{\partial z}{\partial x}=y^2+2xy$ and $\frac{\partial z}{\partial y}=2xy+x^2$

Also, $x=at^2, y=2at \Rightarrow \frac{dx}{dt}=2at, \frac{dy}{dt}=2a$

Hence, $$\frac{dz}{dt}=\frac{\partial z}{\partial x}.\frac{dx}{dt}+\frac{\partial z}{\partial y}.\frac{dy}{dt}=\left(y^2+2xy\right)(2at)+\left(2xy+x^2\right)2a$$

$$=\left(4a^2t^2+2at^2.2at\right)2at+\left(2at^2.2at+a^2t^4\right)2a$$

$$=8a^3t^3+8a^3t^4+8a^3t^3+2a^4t^3=16a^3t^3+8a^3t^4+2a^4t^3$$

Example 16: If $u=4x^2-y^2,\ x=e^t\cos t,\ y=2e^t\sin t$, find $\dfrac{du}{dt}$

Soln: We have, $u=4x^2-y^2 \Rightarrow \dfrac{\partial u}{\partial x}=8x$ and $\dfrac{\partial u}{\partial y}=-2y$

Also $x=e^t\cot t, \quad y=2e^t\sin t$

Hence, $\dfrac{dx}{dt}=e^t\cos t-e^t\cdot\sin t,\ \dfrac{dy}{dt}=2e^t\sin t+2e^t\cos t$

Therefore, $\dfrac{du}{dt}=\dfrac{\partial u}{\partial x}.\dfrac{dx}{dt}+\dfrac{\partial u}{\partial y}.\dfrac{dy}{dt}=8x\left(e^t\cos t-\sin t\,e^t\right)+(-2y).2e^t(\sin t+\cos t)$

$$=8e^t\cos t\left(e^t\cos t-\sin t\,e^t\right)-4e^t\left(2e^t\sin t\right)(\sin t+\cos t)=8e^{2t}(\cos 2t-\sin 2t)$$

Example 17: Find $\dfrac{du}{dx}$ if $u=\sin\left(x^2+y^2\right)$, where $a^2x^2+b^2y^2=c^2$

Soln: We have, $u=\sin\left(x^2+y^2\right) \Rightarrow \dfrac{\partial u}{\partial x}=\cos\left(x^2+y^2\right).2x$ and $\dfrac{\partial u}{\partial y}=\cos\left(x^2+y^2\right).2y$

Given: $a^2x^2+b^2y^2=c^2 \Rightarrow 2a^2x+2b^2y\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{a^2x}{b^2y}$

$$\Rightarrow \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}=2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\left(-\frac{a^2x}{b^2y}\right)=2x\cos\left(x^2+y^2\right)\left(1-\frac{a^2x}{b^2y}\right)$$

Example 18: If $x^y+y^x=c$, find the value of $\dfrac{dy}{dx}$

Soln: Let $f(x,y)=x^y+y^x$

Then, $\dfrac{\partial f}{\partial x}=yx^{y-1}+y^x\log_e y$; Similarly, $\dfrac{\partial f}{\partial y}=x^y\log_e x+xy^{x-1}$

$$\therefore \quad \frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{y^x\log_e y+y^{y-1}}{x^y\log_e x+x\,y^{x-1}}$$

Example 19: Find $\dfrac{d^2y}{dx^2}$ if $\mathrm{x}^4+y^4+4a^2xy=0$

Soln: Let $f(x,y)=x^4+y^4+4a^2xy$

$p=\dfrac{\partial f}{\partial x}=4x^2+4a^2xy$; $q=\dfrac{\partial f}{\partial y}=4x^3+4a^2y$; $r=\dfrac{\partial f}{\partial y}=12x^2$; $s=\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)=4a^2$

and $t=\dfrac{\partial^2 f}{\partial y^2}=12y^2$

$$\frac{d^2y}{dx^2}=-\frac{q^2r-2pqs+p^2t}{q^3}$$

$$=-\frac{\left(4y^3+4a^2x\right)^2.12x^2-2\left(4x^3+4a^2y\right)\left(4y^3+4a^2x\right).4a^2+\left(4x^3+4a^2y\right)^2.12y^2}{\left(4y^3+4a^2x\right)^3}$$

$$=-\frac{\left(3y^3+a^2x\right)^2x^2-2\left(x^3+a^2y\right)\left(y^3+a^2x\right)+\left(x^3+a^2y^2\right)y^2}{\left(y^3+a^2x^2\right)^3}$$

PRACTICE SET

1. If $u=\tan^{-1}\dfrac{xy}{\sqrt{\left(1+x^2+y^2\right)}}$, show that $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{1}{\left(1+x^2+y^2\right)^{3/2}}$

2. If $z=f(x+ay)+\phi(x-ay)$ prove that $\dfrac{\partial^2 z}{\partial y^2}=a^2\dfrac{\partial^2 z}{\partial x^2}$

3. *If* $u=x^2(y-z)+y^2(z-x)+z^2(x-y)$, prove that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$

4. If $u=\log\left(x^2+y^2+z^2\right)$, show that $x\dfrac{\partial^2 u}{dy\partial z}=y\dfrac{\partial^2 u}{\partial z\partial x}=z\dfrac{\partial^2 u}{\partial x\partial y}$

5. If $\mathrm{u}=\tan^{-1}\dfrac{\mathrm{x}+\mathrm{y}}{\mathrm{x}-\mathrm{y}}$, then prove that $\mathrm{x}\dfrac{\partial \mathrm{u}}{\partial \mathrm{x}}+\mathrm{y}\dfrac{\partial \mathrm{u}}{\partial \mathrm{y}}=0$

6. If $z(x+y)=x^2+y^2$, show that $\left(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)^2=4\left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)$

7. If $u=\sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that $\dfrac{\partial u}{\partial x}=-\dfrac{y}{x}\dfrac{\partial u}{\partial y}$

8. If $u=\log_e\sqrt{\left(x^2+y^2+z^2\right)}$, Prove that $\left(x^2+y^2+z^2\right)\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right)=1$

9. If $a^2x^2+b^2y^2-c^2z^2=0$, show that $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=\dfrac{a^2b^2}{c^4z^3}\left(x^2+y^2\right)$

10. If $u=\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, show that $\mathrm{u}_x+u_y+u_z=0$

11. Verify Euler's theorem for the following:

    (a) $u=ax^2+2hxy+by^2$, (b) $u=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$ (c) $u=\dfrac{x\left(x^3-y^3\right)}{x^3+y^3}$

12. If $u=f(y/x)$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

13. If $u=x^2\tan^{-1}\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2u$

14. (a) If $u=\sin^{-1}\dfrac{x+y}{\sqrt{x}-\sqrt{y}}$ show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\tan u$

    (b) If $u=\cos^{-1}\dfrac{x+y}{\sqrt{x}-\sqrt{y}}$ show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=-\dfrac{1}{2}\cot u$

15. (a) If $u = x\phi\left(\frac{y}{x}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

(b) If $u = \sin^{-1}\left(\frac{y}{x}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

16. Find $\frac{dy}{dx}$ (i) $x^y + y^x = (x+y)^{x+y}$ (ii) $(\tan x)^y + y^{\cot x} = a$

17. If $u = x\log xy$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

18. If $ax^2 + by^2 + cz^2 = 1$ and $\ell x + my + nz = 0$, find the values of $\frac{dy}{dx}$ and $\frac{dz}{dx}$

19. If $u = x^2 - y^2 + \sin yz$, where $y = e^x$, and $z = \log_e x$, find $\frac{du}{dx}$

20. If $x = r\cos\theta, y = r\sin\theta$ where r and θ are functions of t, prove that

(a) $\frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}$ (b) $\frac{dy}{dt} = \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}$

(c) $\frac{d^2x}{dt^2} = \cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}.\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}$

(d) $\frac{d^2y}{dt^2} = \sin\theta\frac{d^2r}{dt^2} + 2\cos\theta\frac{dr}{dt}.\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}$

(e) $x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}$

21. $\lim_{x\to 0}\frac{f(x+h,y) - f(x,y)}{h}$, if exists, is called the partial derivatives of f with respect to:

(a) x at (a, b) (b) x at (x, y) (c) y at (a, b) (d) y at (x,y)

22. If $u = \sin^{-1}\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, then :

(a) $\frac{\partial u}{\partial x} = -\frac{x}{y}\frac{du}{dy}$ (b) $\frac{\partial u}{\partial x} = -\frac{x}{y}\frac{\partial u}{\partial x}$ (c) $\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial y}$ (d) $\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial x}$

23. If $f(x,y) = c$, then $\frac{dy}{dx}$ is:

(a) $\frac{\partial f}{\partial x}$ (b) $\frac{\partial f}{\partial y}$ (c) $-\frac{\partial f/dx}{\partial y/\partial y}$ (d) $\frac{\partial f/dx}{\partial y/\partial y}$

24. If $z = \sin(x^2y^2)$, *then* $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 =$

(a) $4x^2y^2\cos^2(x^2y^2)$ (b) $4x^2y^2(x^2+y^2)$

(c) $4x^2y^2(x^2+y^2)\cos^2(x^2y^2)$ (d) None of these.

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25. If $z = (x+y)/(x-y); x \neq y$, then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} =$

(a) $2(x+y)/(x-y)^2$ (b) $2/(x-y)$

(c) $-2/(x-y)$ (d) None of these

26. If $z = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, then $\partial^2 z/\partial x\partial y =$

(a) $(y^2 - x^2)/(x^2 + y^2)$ (b) $(x^2 - y^2)/(x^2 + y^2)^2$

(c) $(x - y^2)/(x^2 + y^2)$ (d) $(x^2 - y^2)/(x^2 + y^2)^2$

27. If $z = \cos(xy^3)$, then $\partial^2 z/\partial x\partial y =$

(a) $-6xy\sin(xy^3) + 9x^2y^4\cos(xy^3)$ (b) $6xy\sin(xy^3) - 9x^2y^4\cos(xy^3)$

(c) $-6xy\sin(xy3) - 9x^2y^4\cos(xy^3)$ (d) $6xy\sin(xy^3) + 9x^2y^4\cos(xy^3)$

28. If $(x,y) = x^3y^3 - 2x^2y^2$, then $(f_{xx})_{x=y=1} =$

(a) 1 (b) -1 (c) $1/x^2$ (d) None of these

29. If $(x,y) = \log(x\tan^{-1}y)$, then $f_{xy} =$

(a) $-1/x^2$ (b) 0 (c) $1/x^2$ (d) None of these

30. If $f(x,y) = \sin(e^{ax} + e^{by})$, then $f_{xy} =$

(a) $-ab\,e^{(ax+by)}\sin(e^{ax} + e^{by})$ (b) $ab(e^{ax} + e^{by})\sin(e^{ax} + e^{by})$

(c) $a^2e^{ax}\left[\cos(e^{ax} + e^{by}) - e^{ax}\sin(e^{ax} + e^{by})\right]$ (d) None of these

31. If $f(x,y,z) = (x^2 + y^2z^2)^{-1/2}$, then $f_{xx} + f_{yy} + f_{zz} =$

(a) 0 (b) 1 (c) -1 (d) None of these

32. If $z = \tan^{-1}(y/x)$, then dz

(a) $\frac{xdy + ydx}{x^2 + y^2}$ (b) $\frac{xdx - ydy}{x^2 + y^2}$ (c) $\frac{xdx - ydy}{x^2 + y^2}$ (d) $\frac{xdy - ydx}{x^2 + y^2}$

33. If $u = f(y-z, z-x, x-y)$, then value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ is:

(a) 3 (b) 0 (c) $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ (d) None of these

34. If $z = e^{ax+by}f(ax-by)$, then

$$\frac{1}{z}\left[b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}\right]$$

(a) ab (b) 2ab (c) ab/2 (d) None of these

35. If u be a homogenous function of degree n, the

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=$$

(a) $n\frac{\partial u}{\partial x}$ (b) $(n-1)\frac{\partial u}{\partial x}$ (c) $(n+1)\frac{\partial u}{\partial x}$ (d) None of these

36. If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=$$

(a) u (b) $\sin u$ (c) $\tan u$ (d) None of these

37. If $z=xy\,f(x/y)$, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$$

(a) z (b) 0 (c) 1/z (d) 2z

38. If $z=\tan^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$$

(a) $\sin z\cos z$ (b) 0 (c) tan z (d) None of these

39. If $\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=z$, then

(a) tan z (b) $\frac{1}{2}\sin z$ (c) $2\tan z$ (d) $\frac{1}{2}\tan z$

40. If $u=\left(x^{1/4}+y^{1/4}\right)/\left(x^{1/6}+y^{1/6}\right)$ and $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=ku$, then $k=$

(a) 1/4 (b) 1/12 (c) 1/24 (d) 1/6

41. If $u=\log\left(x^3+y^3+z^3-3xyz\right)$, then

$(\partial/\partial x+\partial/\partial y+\partial/\partial z)^2 u=\frac{k}{(x+y+z)^2}$, then $k=$

(a) 9 (b) –9 (c) 3 (d) –3

42. If $V=\left(x^2+y^2+z^2\right)^{-1/2}$, then

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=$$

(a) V (b) $-V$ (c) $2V$ (d) None of these

43. $z=f\left(y/x^2\right)$, then $x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial y}=0$

(a) $\frac{4y}{x^2}f'\left(y/x^2\right)$ (b) $\frac{4y}{x^3}f'\left(y/x^2\right)$ (c) 0 (d) None of these

ANSWER KEY

21. (b)	22. (b)	23. (c)	24. (c)	25. (b)	26. (d)	27. (c)
28. (d)	29. (b)	30. (a)	31. (a)	32. (d)	33. (b)	34. (b)
35. (b)	36. (c)	37. (d)	38. (b)	39. (d)	40. (b)	41. (b)
42. (b)	43. (c).					

Chapter 14

TAYLOR'S THEOREM

Taylor's theorem: If $f(x, y)$ is a function which passessses continuous partial derivtives of order n in any domain of a point (a,b), and the domain is large enough to contain a point $(a+h, b+k)$ within it, then there exists a positive number, $(0<\theta<1)$, such that

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\dots$$

$$+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+R_n$$

Where $R_n=\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,b+\theta k),(0<\theta<1)$.

Other form : The theorem can be stated in still another form

$$f(x,y)=f(a,b)+\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a,b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)+\dots$$

$$+\frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b)+R_n$$

where $R_n=\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n f(a+(x-a)\theta,b+(y-b)\theta),0<\theta<1$, called the Taylor's expansion of f(x,y) about the point (a,b) in powers of $x–a$ and $y–b$.

Example 1: Expand $x^2y+3y-2$ in powers of $x-1$ and $y+2$

Soln: Let use Taylor's expansion with a = 1, b = –2. Then

$f(x,y)=x^2y+3y-2,$ $\quad f(1,-2)=-10$

$f_x(x,y)=2xy,$ $\quad f_x(1,-2)=-4$

$f_y(x,y)=x^2+3,$ $\quad f_y(1,-2)=4$

$f_{xx}(x,y)=2y,$ $\quad f_{xx}(1,-2)=-4$

$f_{xy}(x,y)=2x,$ $\quad f_{xy}(1,-2)=2$

$f_{yy}(x,y)=0$ $\quad f_{yy}(1,-2)=0$

$f_{xxx}(x,y)=0 f_{yyy}(x,y)$ $\quad f_{yxx}(1,-2)=2=f_{xxy}(1,-2)$

All higher derivatives are zero,

$$\therefore \quad x^2y+3y-2=-10-4(x-1)+4(y+2)+\frac{1}{2}\left[-4(x-1)^2+4(x-1)(y+2)\right]$$

$$+\frac{1}{3!}3(x-1)^2(y+2)(2)+0$$

$$=-1-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2)$$

Example 2: Expand $f(x,y)=x^2+xy+y^2$ in powers of $(x-2)$ and $(y-3)$

Soln: Here, $f(x,y)=x^2+xy+y^2$

$\therefore \quad f(2,3)=19$

Now, $\frac{\partial f}{\partial x}=2x+y, \frac{\partial f}{\partial y}=x+2y; \frac{\partial^2 f}{\partial x^2}=2; \frac{\partial^2 f}{\partial y^2}=2, \frac{\partial^2 f}{\partial x\partial y}=1$ and the values of third and higher order partial derivatives of f and zero.

Thus at (2, 3) $\frac{\partial f}{\partial x}=7, \frac{\partial f}{\partial y}=8, \frac{\partial^2 f}{\partial x^2}=2, \frac{\partial^2 f}{\partial x\partial y}=1, \frac{\partial^2 f}{\partial y^2}=2$

Hence by Taylor's formula,

$$f(x,y)=f(2,3)+\left[(x-2)\frac{\partial}{\partial x}+(y-3)\frac{\partial}{\partial y}\right]f(2,3)$$

$$+\frac{1}{2!}\left[(x-2)^2\frac{\partial^2}{\partial x^2}+2(x-2)(y-3)\frac{\partial^2}{\partial x\partial y}+(y-3)^2\frac{\partial^2}{\partial y^2}\right]f(2,3)+\dots$$

We have $f(x,y)=19+\left[7(x-2)+8(y-3)\right]$

$$+\frac{1}{2}\left[2(x-2)^2+2(x-2)(y-3)+2(y-3)^2\right]$$

Example 3: Show that $e^{ax}\sin by=by+abxy+\frac{1}{3!}e^{a\theta x}$

$$\left[\left(a^3x^3-3ab^2xy^2\right)\sin b\theta y+\left(3a^2bx^2y+b^2y^3\right)\cos b\theta y\right]$$

where $0<\theta<1$.

Soln: Here by Taylor's theorem,

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)+\frac{1}{2!}\left(x^2\frac{\partial^2}{\partial x^2}+2xy\frac{\partial^2}{\partial x\partial y}+y^2\frac{\partial^2}{\partial y^2}\right)f(0,0)$$

$$+\frac{1}{3!}\left(x^3\frac{\partial^3}{\partial x^3}+3x^2y\frac{\partial^3}{\partial x^2\partial y}+3xy^2\frac{\partial^3}{\partial x\partial y^2}+y^3\frac{\partial^3}{\partial y^3}\right)f(\theta x,\theta y)$$

where $0<\theta<1$

here $f(x,y)=e^{ax}\sin by \therefore f(0,0)=0$

Again $\frac{\partial f}{\partial x} = ae^{ax}\sin by, \frac{\partial f}{\partial y} = be^{ax}\cos by;$

$\frac{\partial^2 f}{\partial x\partial y} = abe^{ax}\cos by, \frac{\partial^2 f}{\partial y^2} = -b^2e^{ax}\sin by\,;\ \frac{\partial^3 \mathrm{f}}{\partial \mathrm{x}^3} = \mathrm{a}^3\mathrm{e}^{\mathrm{ax}}\sin \mathrm{by}, \frac{\partial^3 f}{\partial x\partial y} = a^2be^{ax}\cos by,$

$\frac{\partial^3 f}{\partial x\partial y^2} - ab^2e^{ax}\sin by\,;\ \frac{\partial^3 f}{\partial y^3} = -b^3e^{ax}\cos by$

at (0, 0) $\Rightarrow \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = b;\ \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0, \frac{\partial^2 f}{\partial x\partial y} = ab;$

On substituting θx, in place of x and θy in place of y,

$\frac{\partial^3 f}{\partial x^3} = a^3e^{a\theta x}\sin b\,\theta y, \qquad \frac{\partial^3 f}{\partial x^2\partial y} = a^2be^{a\theta x}\cos b\,\theta y,$

$\frac{\partial^2 f}{\partial x\partial y^2} = -ab^2\, e^{a\theta x}\sin b\,\theta y, \quad \frac{\partial^3 f}{\partial y^3} = -b^3\, e^{a\theta x}\cos b\,\theta y$

$$e^{ax}\sin bx = 0 + \{0 + by\} + (1/2!)\{0 + 2xy(ab) + (0)\}$$
$$+(1/3!)e^{a\theta x}\{a^3x^3 - 3ab^2xy^2\sin b\,\theta y + (3a^2bx^2y - b^3y^3)\cos b\,\theta y\}$$

$$\Rightarrow e^{ax}\sin bx = by + abxy + (1/3!)e^{a\theta x}\{(a^3x^3 - 3ab^2xy^2)\sin b\,\theta y + (3a^2bx^2y - b^3y^3)\cos b\,\theta y\}$$

Chapter 15

JACOBIAN

Definition: If $u_1, u_2, u_3, \ldots\ldots, u_n$ be the functions of n variables $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ then the determinant.

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots. u_n$ with respect to $x_1, x_2, x_3, \ldots. x_n$. We shall generally denote the Jacobain of $u_1, u_2, u_3, \ldots, u_n$ by

$$\frac{\partial(u_1, u_2, u_3, \ldots\ldots\ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots\ldots, x_n)} \text{ or } \mathrm{J}(u_1, u_2, u_3, \ldots. u_n)$$

Particular case :

If functions $u_1, u_2, \ldots\ldots, u_n$ of $\mathrm{x}_1, x_2, \ldots. x_n$ are of the form

$$u_1 = f_1(x_1);\ u_2 = f_2(x_1, x_2), \ldots, u_n = f_n(x_1, x_2, \ldots. x_n)$$

then it is clearly seen that

$$\frac{\partial u_1}{\partial x_2} = 0 = \frac{\partial u_1}{\partial x_3} = \ldots. = \frac{\partial u_1}{\partial x_n} \text{ and } \frac{\partial u_2}{\partial x_3} = 0 = \frac{\partial u_2}{\partial x_4} = \ldots = \frac{\partial u_2}{\partial x_n};\ etc.$$

and therefore, $$\frac{\partial(u_1, u_2, u_3, \ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots\ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = \frac{\partial u_1}{\partial x_1}.\frac{\partial u_2}{\partial x_2}.\frac{\partial u_3}{\partial x_3}\ldots\frac{\partial u_n}{\partial x_n}$$

i.e, the Jacobain reduces to its leading terms.

Example 1: If $x=r\cos\theta,\ y=r\sin\theta$, then find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$ and $\dfrac{\partial(r,\theta)}{\partial(x,y)}$

Soln: $$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\left(\cos^2\theta+\sin^2\theta\right)=r$$

Again we have, $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\dfrac{y}{x}$

$$\frac{\partial r}{\partial x}=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2x=\cos\theta\,; \qquad \frac{\partial r}{\partial y}=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2y=\sin\theta\,;$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}.\left(-\frac{y}{x^2}\right)=-\frac{\sin\theta}{r}\,; \qquad \frac{\partial\theta}{\partial y}=\frac{\cos\theta}{r}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y}\end{vmatrix}=\begin{vmatrix}\cos\theta & \sin\theta\\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r}\end{vmatrix}=1/r\left(\cos^2\theta+\sin^2\theta\right)=1/r$$

Example 2: If $u_1=\dfrac{x_2x_3}{x_1},\ u_2=\dfrac{x_3x_1}{x_2}$ and $u_3=\dfrac{x_1x_2}{x_3}$, then prove that, $J\left(u_1,u_2,u_3\right)=4$

Soln: We have $$J\left(u_1,u_2,u_3\right)=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{vmatrix}=\begin{vmatrix}-\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1}\\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2}\\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2}\end{vmatrix}$$

$$=\frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix}-x_2x_3 & x_3x_1 & x_1x_2\\ x_2x_3 & -x_3x_1 & x_1x_2\\ x_2x_3 & x_3x_1 & x_1x_2\end{vmatrix}=\frac{\left(x_2x_3\right)\left(x_3x_1\right)\left(x_1x_2\right)}{x_1^2x_2^2x_3^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4$$

Example 3: *If* $y_1=\left(1-x_1\right),\ y_2=x_1\left(1-x_2\right),\ y_3=x_1x_2\left(1-x_3\right),\ldots.$

$y_n=x_1x_2\ldots x_{n-1}\left(1-x_n\right)$, then prove that $\dfrac{\partial\left(y_1,y_2,\ldots,y_n\right)}{\partial\left(x_1,x_2,\ldots\ldots x_n\right)}=(-1)^n x_1^{n-1}x_2^{n-2}\ldots\ldots x_{n-1}$

Soln: $$\frac{\partial\left(y_1,y_2,\ldots\ldots,y_n\right)}{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}=\frac{\partial y_1}{\partial x_1}.\frac{\partial y_2}{\partial x_2}\ldots\ldots\frac{\partial y_n}{\partial x_n}=(-1).\left(-x_1\right).\left(-x_1x_2\right)\ldots\ldots\left(-x_1x_2\ldots\ldots x_{n-1}\right)$$

$$=(-1)^n x_1^{n-1}.x_2^{n-2}\ldots\ldots x_{n-1}$$

Jacobian of Implicit functions:

If $u_1,u_2,u_3,\ldots\ldots u_n$ and $x_1,x_2,\ldots\ldots\ldots x_n$ are implicitly connected by n equations as:

$$f_1\left(u_1,u_2,\ldots\ldots\ldots u_n,x_1,x_2,\ldots\ldots,x_n\right)=0$$
$$f_2\left(u_1,u_2,\ldots\ldots,u_n,x_1,x_2,\ldots\ldots x_n\right)=0$$

....

....

$$f_n\left(u_1,u_2,\ldots\ldots,u_n,x_1,x_2,\ldots\ldots,x_n\right)=0$$

then $$\frac{\partial\left(f_1,f_2,\ldots\ldots f_n\right)}{\partial\left(u_1,u_2,\ldots\ldots u_n\right)}.\frac{\partial\left(u_1,u_2,\ldots\ldots,u_n\right)}{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}=(-1)^n\frac{\partial\left(f_1,f_2,\ldots\ldots,f_n\right)}{\partial\left(x_1,x_2,\ldots\ldots x_n\right)}$$

Note : The above result is a generalised result of the elementary theorem $\dfrac{dy}{dx}=-\dfrac{\partial f/\partial x}{\partial f/\partial y}$

Jacobian of function of function :

Theorem : If $u_1,u_2,u_3,\ldots\ldots\ldots u_n$ are functions of $y_1,y_2,y_3,\ldots,y_n$ and $y_1,y_2,y_3,\ldots,y_n$ are functions of $x_1,x_2,x_3,\ldots\ldots x_n$ then,

$$\frac{\partial\left(u_1,u_2,\ldots\ldots,u_n\right)}{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}=\frac{\partial\left(u_1,u_2,\ldots\ldots,u_n\right)}{\partial\left(y_1,y_2,\ldots\ldots,y_n\right)}.\frac{\partial\left(y_1,y_2,\ldots\ldots,y_n\right)}{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}$$

Cor: If we take $u_1=x_1,\ u_2=x_2,\ldots,u_n=x_n$ then the above result becomes

$$\frac{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}{\partial\left(y_1,y_2,\ldots,y_n\right)}\times\frac{\partial\left(y_1,y_2,\ldots\ldots,y_n\right)}{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}=1$$

Particular cases : If the implicit relations in .... are given are

$$f_1\left(x_1,x_2,\ldots\ldots,x_n,u_1\right)=0$$
$$f_2\left(x_2,x_3,\ldots\ldots,x_n,u_1,u_2\right)=0$$
$$f_3\left(x_3,x_4,\ldots\ldots,x_n,u_1,u_2,u_3\right)=0$$

....

$$f_n\left(x_n,u_1,u_2,\ldots\ldots,u_n\right)=0$$

then it is easily seen that

$$\frac{\partial\left(f_1,f_2,\ldots\ldots\ldots,f_n\right)}{\partial\left(u_1,u_2,\ldots\ldots,u_n\right)}=\frac{\partial f_1}{\partial u_1}.\frac{\partial f_2}{\partial u_2}\ldots\ldots\frac{\partial f_n}{\partial u_n}$$

and $$\frac{\partial\left(f_1,f_2,\ldots\ldots,f_n\right)}{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}=\frac{\partial f_1}{\partial x_1}.\frac{\partial f_2}{\partial x_2}\ldots\frac{\partial f_n}{\partial x_n}$$

$$\therefore \quad \frac{\partial\left(u_1,u_2,\ldots\ldots,u_n\right)}{\partial\left(x_1,x_2,\ldots\ldots,x_n\right)}=(-1)^n\frac{\frac{\partial f_1}{\partial x_1},\frac{\partial f_2}{\partial x_2}\ldots\frac{\partial f_n}{\partial x_n}}{\frac{\partial f_1}{\partial u_1}.\frac{\partial f_2}{\partial u_2}\ldots\frac{\partial f_n}{\partial u_n}}$$

Example 4: If $x+y+z=u,\ y+z=uv,\ z=u\,v\,w$, find the value of the Jacobian of x, y, z with respect to u, v, w.

Soln: Let,
$$f_1=x+y+z-u=0$$
$$f_2=y+z-uv=0$$
$$f_3=z-uvw=0$$

Then, $$\frac{\partial(x,y,z)}{\partial(u,v,w)}=(-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\div\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

$$=-\begin{vmatrix}\frac{\partial f_1}{\partial u}&\frac{\partial f_1}{\partial v}&\frac{\partial f_1}{\partial w}\\\frac{\partial f_2}{\partial u}&\frac{\partial f_2}{\partial v}&\frac{\partial f_2}{\partial w}\\\frac{\partial f_3}{\partial u}&\frac{\partial f_3}{\partial v}&\frac{\partial f_3}{\partial w}\end{vmatrix}\div\begin{vmatrix}\frac{\partial f_1}{\partial x}&\frac{\partial f_1}{\partial y}&\frac{\partial f_1}{\partial z}\\\frac{\partial f_2}{\partial x}&\frac{\partial f_2}{\partial y}&\frac{\partial f_2}{\partial z}\\\frac{\partial f_3}{\partial x}&\frac{\partial f_3}{\partial y}&\frac{\partial f_3}{\partial z}\end{vmatrix}=-\begin{vmatrix}-1&0&0\\-v&-u&0\\-vw&-uw&-uv\end{vmatrix}\div\begin{vmatrix}1&1&1\\0&1&1\\0&0&1\end{vmatrix}$$

$$=-(-1)(-u)(-uv)\div(1)(1)(1)=\mathrm{u}^2\mathrm{v}$$

Example 5: Given the roots of the equation in λ,

$(\lambda-\mathrm{x})^3+(\lambda-\mathrm{y})^3+(\lambda-\mathrm{z})^3=$ are u,v, w then prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=-2\frac{(y-z)(z-w)(x-y)}{(v-w)(\omega-u)(u-v)}$$

Soln: Equation in λ can be written as
$$3\lambda^3-3\lambda^2(x+y+z)+3\lambda\left(x^2+y^2+z^2\right)-\left(x^3+y^3+z^3\right)=0$$
If the roots of the above equation be u, v, w then
$$\Sigma u=u+v+w=x+y+z,;\ \Sigma uv=uv+vw+wu=x^2+y^2+z^2$$
and $$uvw=\frac{1}{3}\left(x^3+y^3+z^3\right)$$
these relations can now be written as
$$f_1=u+v+w-x-y-z=0;\ f_2=uv+vw+wu-x^2-y^2-z^2=0$$
$$\mathrm{f}_3=\mathrm{uvw}-\frac{1}{3}\left(\mathrm{x}^3+\mathrm{y}^3+\mathrm{z}^3\right)=0$$
$$\therefore\quad\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\div\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$

$$=-\begin{vmatrix}\frac{\partial f_1}{\partial x}&\frac{\partial f_1}{\partial y}&\frac{\partial f_1}{\partial z}\\\frac{\partial f_2}{\partial x}&\frac{\partial f_2}{\partial y}&\frac{\partial f_2}{\partial z}\\\frac{\partial f_3}{\partial x}&\frac{\partial f_3}{\partial y}&\frac{\partial f_3}{\partial z}\end{vmatrix}\div\begin{vmatrix}\frac{\partial f_1}{\partial u}&\frac{\partial f_1}{\partial v}&\frac{\partial f_1}{\partial w}\\\frac{\partial f_2}{\partial u}&\frac{\partial f_2}{\partial v}&\frac{\partial f_2}{\partial w}\\\frac{\partial f_3}{\partial u}&\frac{\partial f_3}{\partial v}&\frac{\partial f_3}{\partial w}\end{vmatrix}=-\begin{vmatrix}-1&-1&-1\\-2x&-2y&-2z\\-x^2&-y^2&-z^2\end{vmatrix}\div\begin{vmatrix}1&1&1\\v+w&w+u&u-w\\vw&uw&uv\end{vmatrix}$$

$$=\begin{vmatrix}-1&0&0\\-2x&2(x-y)&2(x-z)\\x^2&x^2-y^2&x^2-z^2\end{vmatrix}/\begin{vmatrix}1&1&1\\v+w&u-v&u-w\\vw&w(u-v)&v(u-w)\end{vmatrix}=-2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Example 6: If $u=x\left(1-r^2\right)^{-1/2},\ v=y\left(1-r^2\right)^{-1/2},\ w=z\left(1-r^2\right)^{-1/2}$ where $r^2=x^2+y^2+z^2$, show that $f(u,v,w)=\left(1-r^2\right)^{-5/2}$

Soln: Given, $u=x\left(1-r^2\right)^{-1/2}\Rightarrow x^2=u^2\left(1-x^2-y^2-z^2\right)$ etc.

$\therefore$ Let $\mathrm{f}_1\equiv\mathrm{x}^2-\mathrm{u}^2\left(1-\mathrm{x}^2-\mathrm{y}^2-\mathrm{z}^2\right)=0;\ f_2\equiv y^2-v^2\left(1-x^2-y^2-z^2\right)=0$

and $f_3\equiv z^2-w^2\left(1-x^2-y^2-z^2\right)=0$

$$\therefore\quad\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3.\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\div\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$

Now, $$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}=\begin{vmatrix}2x\left(1+u^2\right)&2yu^2&2zu^2\\2xv^2&2y\left(1+v^2\right)&2zv^2\\2xw^2&2yw^2&2z\left(1+w^2\right)\end{vmatrix}=8xyz\begin{vmatrix}1+u^2&u^2&u^2\\v^2&1+v^2&v^2\\w^2&w^2&1+w^2\end{vmatrix}.$$
$$=8xyz\left(1+u^2+v^2+w^2\right)$$

Also, $$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}=\begin{vmatrix}-2u\left(1-r^2\right)&0&0\\0&-2v\left(1-r^2\right)&0\\0&0&-2\omega\left(1-r^2\right)\end{vmatrix}=-8uvw\left(1-r^2\right)^3$$

$$\therefore\quad\frac{\partial(u,v,w)}{\partial(u,v,w)}=\frac{8xyz\left(1+u^2+v^2+w^2\right)}{-8uvw\left(1-r^2\right)^3}=\left(1-r^2\right)^{-5/2}$$

as $$1+u^2+v^2+w^2+\frac{r^2}{1-r^2}=\frac{1}{1-r^2}\ \text{and}\ uvw=xyz\left(1-r^2\right)^{-3/2}$$

Example 7: If $u^3+v^3+w^3=x+y+z,\ u^2+v^2+w^2=x^3+y^3+z^3,\ u+v+w=x^2+y^2+z^2$, then show that

Soln: Let, $f_1\equiv u^3+v^3+w^3-x-y-z=0,\ f_2\equiv u^2+v^2+w^2-x^3-y^3-z^3=0$

and $f_3\equiv u+v+w-x^2-y^2-z^2=0$

We have, $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3.\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\div\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$

$$=-\begin{vmatrix}\frac{\partial \mathrm{f}_1}{\partial \mathrm{x}}&\frac{\partial \mathrm{f}_1}{\partial \mathrm{y}}&\frac{\partial \mathrm{f}_1}{\partial \mathrm{z}}\\\frac{\partial \mathrm{f}_2}{\partial \mathrm{x}}&\frac{\partial \mathrm{f}_2}{\partial \mathrm{y}}&\frac{\partial \mathrm{f}_2}{\partial \mathrm{z}}\\\frac{\partial \mathrm{f}_3}{\partial \mathrm{x}}&\frac{\partial \mathrm{f}_3}{\partial \mathrm{y}}&\frac{\partial \mathrm{f}_3}{\partial \mathrm{z}}\end{vmatrix}\div\begin{vmatrix}\frac{\partial \mathrm{f}_1}{\partial \mathrm{u}}&\frac{\partial \mathrm{f}_1}{\partial \mathrm{v}}&\frac{\partial \mathrm{f}_1}{\partial \mathrm{w}}\\\frac{\partial \mathrm{f}_2}{\partial \mathrm{u}}&\frac{\partial \mathrm{f}_2}{\partial \mathrm{v}}&\frac{\partial \mathrm{f}_2}{\partial \mathrm{w}}\\\frac{\partial \mathrm{f}_3}{\partial \mathrm{u}}&\frac{\partial \mathrm{f}_3}{\partial \mathrm{v}}&\frac{\partial \mathrm{f}_3}{\partial \mathrm{w}}\end{vmatrix}=-\begin{vmatrix}-1&-1&-1\\-3\mathrm{x}^2&-3\mathrm{y}^2&-3\mathrm{z}^2\\-2\mathrm{x}&-2\mathrm{y}&-2\mathrm{z}\end{vmatrix}\div\begin{vmatrix}3\mathrm{u}^2&3\mathrm{v}^2&3\mathrm{w}^2\\2\mathrm{u}&2\mathrm{b}&2\mathrm{w}\\1&1&1\end{vmatrix}$$

$$=6\begin{vmatrix}1 & 1 & 1\\ x^2 & y^2 & z^2\\ x & y & z\end{vmatrix}\div 6\begin{vmatrix}u^2 & v^2 & w^2\\ u & v & w\\ 1 & 1 & 1\end{vmatrix}=\frac{(y-z)(z-x)(x-y)}{(u-v)(u-w)(w-u)}$$

Example 8: If λ,μ are the roots of the equation in k, $\frac{x}{a+k}+\frac{y}{b+k}+\frac{z}{c+k}=1$, prove that

$$\frac{\partial(x,y,z)}{\partial(\lambda,\mu,v)}=-\frac{(\mu-v)(v-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}$$

Soln: The given equation can be written as

$$(a+k)(b+k)(c+k)-x(b+k)(c+k)-y(c+k)(a+k)-z(a+k)(b+k)=0$$

Or, $k^3+k^2(a+b+c-x-y-z)+k\{ab+bc+ca-(b+c)x-(c+a)y-a(a+b)z\}$

$$+abc-bcx-cay-abz=0$$

Since the roots of this equation are λ,μ,v we have

$$\lambda+\mu+v=x+y+z-a-b-c$$

$$\lambda\mu+\mu v+v\lambda=ab+bc+ca-(b+c)x-(c+a)y-(a+b)z$$

and $\lambda\mu v=bcx+cay+abz-abc$

Let $f_1\equiv\lambda+\mu+v-x-y-z+a+b+c$

$$f_2\equiv\lambda\mu+\mu v+v\lambda+(b+c)x+(c+a)y+(a+b)z-ab-bc-ca$$

and $f_3\equiv\lambda\mu v+bcx-cay-abz+abc$

then $\frac{\partial(x,y,z)}{\partial(\lambda,\mu,v)}=(-1)^3\cdot\frac{\partial(f_1,f_2,f_3)}{\partial(\lambda,\mu,v)}\div\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$

$$=-\begin{vmatrix}1 & 1 & 1\\ \mu+v & v+\lambda & \lambda+\mu\\ \mu v & v\lambda & \lambda v\end{vmatrix}\div\begin{vmatrix}-1 & -1 & -1\\ b+c & c+a & a+c\\ -bc & -ca & -ab\end{vmatrix}=-\frac{(\mu-v)(v-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}$$

Theorem : Let $u_1,u_2,\ldots\ldots,u_n$ be functions of n independent variables $x_1,x_2,\ldots\ldots x_n$. The necessary and sufficient condition that the function be connected by a relation $f(u_1,u_2,\ldots\ldots\ldots,u_n)=0$ is that the Jacobian $\frac{\partial(u_1,u_2,\ldots\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}$ vanishes identically.

Example 9: Show that the functions: $u=3x+2y-z$, $v=x-2y+z$ and $w=x(x+2y-z)$ are not independent, and find the relation between them.

Soln: If the given functions u, v, w are not independent, then we must have $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ equal to zero, or

$$J=\begin{vmatrix}3 & 2 & 1\\ 1 & -2 & -1\\ (2x+2y-z) & 2x & -x\end{vmatrix}=0$$

Adding R_2 to R_1, we get

Since, $J=0$, hence the functions are not independent. Now we have to find the relation between u, v, w.

Clearly, $u+v=4x$ and $u-v=2(x+2y-z)$

$$\therefore\quad (u+v).(u-v)=8x(x+2y-z)=8w\Rightarrow u^2-v^2=8w$$

Example 10: If $u=\frac{x+y}{z}$, $v=\frac{y+z}{x}$, $w=\frac{y(x+y+z)}{xz}$, show that u, v, w are not independent and find the relation between them.

Soln: We have

$$J=\begin{vmatrix}\frac{1}{z} & \frac{1}{z} & -\frac{(x+y)}{z^2}\\ -\frac{(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x}\\ -\frac{y^2-yz}{x^2z} & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2}\end{vmatrix}$$

taking, $\frac{1}{z^2},\frac{1}{x^2}$ and $\frac{1}{x^2z^2}$ common from R_1,R_2 and R_3 respectively, we get

$$J=\frac{1}{x^4z^4}\begin{vmatrix}z & z & -(x+y)\\ -(y+z) & x & x\\ -(y^2z+yz^2) & \frac{x+2y+z}{xz} & \frac{-xy-y^2}{xz^2}\end{vmatrix}$$

taking $\frac{1}{z^2},\frac{1}{x^2}$ and $\frac{1}{x^2z^2}$ common from R_1, R_2 and R_3 respectively, we get

$$J=\frac{1}{x^4z^4}\begin{vmatrix}z & z & -(x+y)\\ -(y+z) & x & x\quad z\\ -(y^2z+yz^2) & x^2z+2xyz+xz^2 & -(x^2y+xy^2)\end{vmatrix}$$

Applying C_2-C_1 and C_3-C_1,

$$=\frac{1}{x^4z^4}\begin{vmatrix}z & 0 & -(x+y+z)\\ -(y+z) & (x+y+z) & (x+y+z)\\ -yz(y+z) & z(x^2+y^2+2xy)+z^2(x+y) & -xy(x+y)+yz(y+z)\end{vmatrix}$$

$$=\frac{1}{x^4z^4}\begin{vmatrix}z & 0 & -(x+y+z)\\ -(y+z) & (x+y+z) & (x+y+z)\\ -yz(y+z) & z(x+y)(z+y+z) & (x+y+z)(yz-xy)\end{vmatrix}$$

$$=\frac{(x+y+z)^2}{x^4y^4}\begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & (yz-xy) \end{vmatrix}$$

Applying C_1+zC_3, we get

$$=\frac{(x+y+z)^2}{x^4z^4}(-1)\begin{vmatrix} -y & 1 \\ -yz(y+x) & z(x+y) \end{vmatrix}=\frac{(x+y+z)^2}{x^4z^4}\left[-yz(x+y)+yz(x+y)\right]=0$$

Since $J=0$, hence the given functions are not independent.

Again. $uv=\frac{xy+y^2+yz+zx}{zx}=\frac{y(x+y+z)}{xz}+1=w+1$

$\therefore$ $uv=w+1$ is the required relation between them.

PRACTICE SET

1. If each of u, v, w is a function of the variables $u-v=w$ then the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ is a determinant of order:
   (a) 9 (b) 3 (c) 1 (d) n

2. If $x=r\cos\theta$, $y=r\sin\theta$, then $\frac{\partial(x,y)}{\partial(r,\theta)}$ is equal to:
   (a) x (b) y (c) r (d) θ

3. If $x=r\cos\theta$, $x=r\cos\theta$, then $\frac{\partial(r,\theta)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(r,\theta)}$ is equal to:
   (a) 0 (b) 1 (c) 2 (d) ∞

4. If $u=f_1(x_1), u_2=f_2(x_1,x_2).....,u_n=f_n(x_1,x_2,....,x_n)$, then $\frac{\partial(u_1,u_2,........,u_n)}{\partial(x_1,x_2,......,x_n)}$ is equal :
   (a) $\frac{\partial u_1}{\partial x_1}$ (b) $\frac{\partial u_1}{\partial x_1}.\frac{\partial u_n}{\partial x_n}$ (c) $\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}....\frac{\partial u}{\partial x_n}$ (d) None of these

5. If $x=u(1+u)$, $y=v(1+u)$, then the Jacobian of x,y with respect to u, v is:
   (a) $u+v$ (b) $1+u+v$ (c) $1-u+v$ (d) $1-u-v$

6. $\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=$
   (a) 1 (b) −1 (c) 0 (d) None of these

Answer Key

1. (b) 2. (c) 3. (b) 4. (c) 5. (b) 6. (a)

Chapter 16

MULTIPLE INTEGRALS

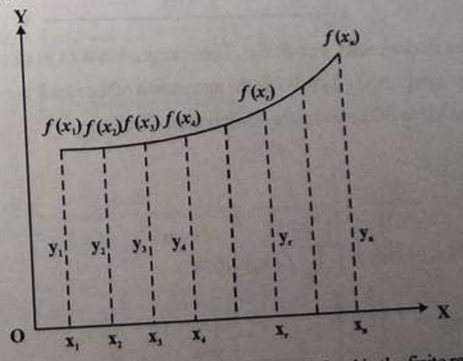
Multiple Integrals

The process of integration for one variable can be extended to the functions of more than one variable. The generalization of definite integrals is known as multiple integral.

Double Integration

We know that

$$\int_a^b f(x)\,dx=\lim_{\substack{n\to\infty\\ \delta x\to 0}}\left[f(x_1)\delta x_1+f(x_2)\delta(x_2)+f(x_3)\delta x_3+.....f(x_n)\delta x_n\right]$$



Let us consider a function $f(x,y)$ of two variable x and y defined in the finite region A of xy-plane. Divide the region A into elementary areas, $\delta A_1, \delta A_2, \delta A_3, ...\delta A_n$

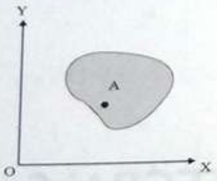
Then $\iint_A f(x,y)\,dA=\lim_{\substack{n\to\infty\\ \delta A\to 0}}\left[f(x_1,y_1)\delta A_1+f(x_2,y_2)\delta A_2+.....+f(x_n,y_n)\delta A_n\right]$

Evaluation of Double Integral

Double integral over region A may be evaluated by two successive integrations.

If A is described as $f_1(x)\le y\le f_2(x)[y_1\le y\le y_2]$ and $a\le x\le b$

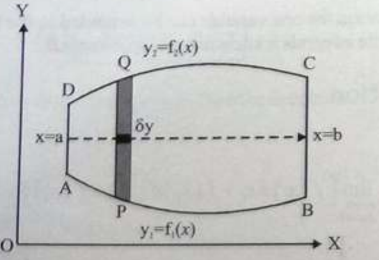
Then, $\iint_A f(x,y)\,dA=\int_a^b\int_{y_1}^{y_2} f(x,y)\,dxdy$



Method 1:

$$\iint_A f(x,y)\,dA = \int_a^b \int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

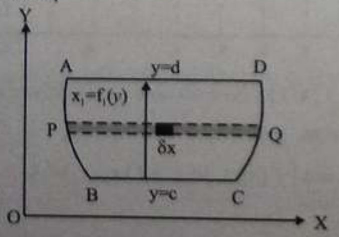
$f(x,y)$ is first integrated with respect to y treating x as constant between the limits a and b.



In the region we take an elementary area $\delta x\delta y$. Then integration w.r.t y (x keeping cosntant), converts small rectangle $\delta x\delta y$ into a strip $PQ(y\delta x)$. While the integration of the result w.r.t. x corresponding to the sliding to the strip PQ, from AD to BC covering the while region ABCD.

Method 2:

$$\iint_A f(x,y)\,dxdy = \int_c^d \int_{x_1}^{x_2} f(x,y)\,dxdy$$



Here $f(x,y)$ is first integrated w.r.t x keeping y constant between the limits x_1 and x_2 and then the resulting expression is integrated with respect to y between the limits c and d.

Take a small area $\delta x\delta y$. The integration w.r.t. x between the limits x_1, x_2 keeping y fixed indicates that integration is done, along PQ. Then the integration of result w.r.t. y corresponds to sliding the strips PQ from BC to AD covering the whole region ABCD.

Note: For constant limits, it does not matter whether we first integrate w.r.t x and then w.r.t y or vice versa.

Example 1: Evaluate $\int_0^{\pi}\int_0^{x} \sin y\,dydx$

Soln: We have, $I = \int_0^{\pi}\left[\int_0^{x} \sin y\,dy\right]dx = \int_0^{\pi}\left[-\cos y\right]_0^{x} dx$, treating x as constant

$$= -\int_0^{\pi}(\cos x - \cos 0)\,dx = -\int_0^{\pi}\cos dx + \int_0^{\pi} dx = -\left[\sin x\right]_0^{\pi} + \left[x\right]_0^{\pi} = \pi$$

Example 2: Evaluate $\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dxdy$

Soln: We have $I = \int_1^{\log 8}\left[\int_0^{\log y}\left(e^x dx\right)\right]e^y dy = \int_0^{\log 8}\left[e^x\right]_0^{\log y} e^y dy$

$$= \int_1^{\log 8} e^y\left[e^{\log y} - e^0\right]dy = \int_1^{\log 8} e^y(y-1)\,dy$$

$$= \left[e^y.y - 2e^y\right]_1^{\log 8} = e^{\log 8}.\log 8 - e - 2e^{\log 8} + 2e = 8\log 8 - 16 + \mathrm{e}$$

Example 3: Evaluate $\int_1^{\sqrt{2}}\int_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} y\,dxdy$

Soln: We have $I = \int_0^{\sqrt{2}}\left[\int_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} dx\right]ydy = \int_0^{\sqrt{2}}\left[x\right]_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} y\,dy = \int_0^{\sqrt{2}} 2\sqrt{(4-2y^2)}\,y\,dy = -\frac{1}{2}\int_4^0 \sqrt{t}\,dt$

by putting $4-2y^2 = t, -4ydy = dt$; $1 = -\frac{1}{2}.\frac{2}{3}\left[t^{3/2}\right]_4^0 = \frac{1}{3}.8 = \frac{8}{3}$

Example 4: Evaluate : $\int_0^1\int_0^1 \frac{dxdy}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}}$

Soln: The given integral $= \int_0^1\int_0^1 \frac{dxdy}{\sqrt{(1-x^2)}\sqrt{(1-y^2)}} = \int_1^0 \frac{1}{\sqrt{(1-y^2)}}\left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)}}\right]dx$

$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}}\left[\sin^{-1} x\right]_0^1 dy = \frac{\pi}{2}\int_0^1 \frac{dy}{\sqrt{(1-y^2)}} = \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{4}$$

Example 5: Solve : $\int_0^1 dx\int_0^{\sqrt{(1-x^2)}} \sqrt{(1-x^2-y^2)}\,dy$

Soln: Then $I = \int_0^1 \frac{1}{2}\left[y\sqrt{(1-x^2-y^2)} + (1-x^2)\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right]_0^{\sqrt{(1-x^2)}} dx$

$$= \frac{1}{2}\int_0^1\left[0 + (1-x^2)\sin^{-1}(1)\right]dx = \frac{\pi}{4}\int_0^1 (1-x^2)\,dx = \frac{\pi}{4}\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{\pi}{4}\left[\left(1-\frac{1}{3}\right) - (0-0)\right] = \frac{\pi}{6}$$

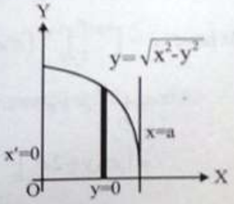
Example 6: Evaluate $\iint_A xy\,dx\,dy$ over the positive quadrant of circle $x^2+y^2=a^2$

Soln: Here the region of integration is positive quadrant of circle $x^2+y^2=a^2$, where x varies from 0 to a and y varies from 0 to $\sqrt{(a^2-x^2)}$.

Hence, $\iint_A xy\,dx\,dy$

$$=\int_0^a\int_0^{\sqrt{(a^2-x^2)}} xy\,dx\,dy$$

$$=\int_0^a\left[\int_0^{\sqrt{(a^2-x^2)}} y\,dy\right]x\,dx$$



$$=\int_0^a\left[\frac{y^2}{2}\right]_0^{\sqrt{(a^2-x^2)}} xdx=\frac{1}{2}\int_0^a x\left(a^2-x^2\right)dx=\frac{1}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a=\frac{1}{8}a^4$$

Example 7: Evaluate $\iint xy(x+y)dx\,dy$ over the area between $y=x^2$ and $y=x$

Soln: The area is bounded by the curves.

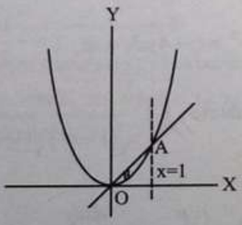
$$y=f_1(x)=x^2,\ y=f_2(x)=x$$

When $f_1(x)=f_2(x), x^2=x$

$\Rightarrow x(x-1)=0\Rightarrow x=0, x=1$

The area of integration is bounded by

$y=x^2, y=x, x=0, x=1$



$\therefore \quad \iint_A xy(x+y)dxdy$

$$=\int_0^1\left[\int_{x^2}^x xy(x+y)dy\right]dx=\int_0^1\left[\int_{x^2}^x\left(x^2y+xy^2\right)dy\right]dx$$

$$=\int_0^1\left[\int_{x^2}^x\left(x^2y+xy^2\right)dx\right]dx=\int\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{x^2}^x dx$$

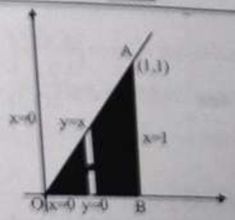
$$=\int_0^1\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx=\left[\frac{x^5}{6}-\frac{x^7}{14}-\frac{x^8}{24}\right]_0^1=\left[\frac{1}{6}-\frac{1}{14}-\frac{1}{24}\right]=\frac{3}{56}$$

Example 8: When the region of integration R is the triangle bounded by y=0, y=x and x=1, show that

$$\iint_R\left(\sqrt{4x^2-y^2}\right)dxdy=\frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

Soln: The region of integration is shown shaded in the adjoining figure. Let us divide the triangle OAB into vertical strips. Then it is evident that in an elementary strip y varies from $y=0$ to $y=x$ while x varies from $x=0$ to $x=1$. Thus the given double integral can be expressed as the repeated integral.

$$\iint_R\sqrt{(4x^2-y^2)}dxdy$$



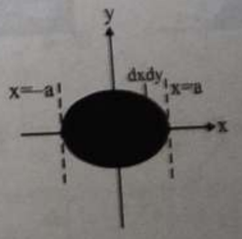
$$=\int_{x=0}^1\int_{y=0}^x\sqrt{(4x^2-y^2)}\,dxdy=\int_{x=0}^1\left[\frac{1}{2}y\sqrt{(4x^2-y^2)}+\frac{1}{2}.4x^2.\sin^{-1}\frac{y}{2x}\right]_{y=0}^x$$

$$=\int_{x=0}^1\frac{1}{2}\left[\sqrt{3}x^2+4x^2.\frac{\pi}{6}\right]dx=\frac{1}{2}\left[\sqrt{3}.\frac{x^3}{3}+\frac{2\pi}{3}.\frac{x^3}{3}\right]_0^1=\frac{1}{2}\left[\sqrt{3}.\frac{1}{3}+\frac{2\pi}{3}.\frac{1}{3}\right]=\frac{1}{3}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right]$$

Example 9: Evaluate $\iint(x+y)^2\,dx\,dy$ over the area bounded by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Soln: For the given ellipse, we have $\frac{y}{b}=\pm\sqrt{\left\{1-\left(\frac{x}{a}\right)^2\right\}}$

Hence the region of integration R can be considered as bounded by the following curves:



$$y=-b\sqrt{\left(1-\frac{x^2}{a^2}\right)},\ y=b\sqrt{\left(1-\frac{x^2}{a^2}\right)};\ \text{x}=-\text{a and x}=\text{a}$$

$$\iint_R(x+y)^2\,dydx=\int_{-a}^a\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}\left(x^2+2xy+y^2\right)dxdy=2\int_{-a}^a\int_0^{b\sqrt{(1-x^2/a^2)}}\left(x^2+y^2\right)dxdy$$

$$=4\int_0^a\int_0^a\left(x^2+y^2\right)dxdy=4\int_0^a\left[x^2y+\frac{1}{3}y^3\right]_0^{b\sqrt{(1-x^2/a^2)}}dx=4\int_0^a\left\{x^2b\left(1-\frac{x^2}{a^2}\right)+\frac{1}{3}b^3\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

$$=4b.\int_0^{\pi/2}\left(a^2\sin^2\theta\cos\theta+\frac{1}{2}b^2\cos^3\theta\right)a\cos\theta d\theta$$

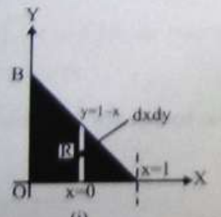
On putting $x=a\sin\theta$, so that $dx=a\cos\theta d\theta$

$$=4b\int_0^{\pi/2}\left\{a^2\sin^2\theta\cos\theta+\frac{1}{3}b^2\cos^2\theta d\theta\right\}=4ab\left\{a^2.\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(3)}+\frac{1}{3}b^2.\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}\right\}$$

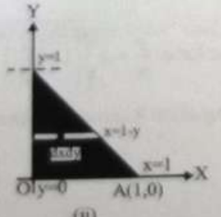
$$=4ab\left\{a^2.\frac{\frac{1}{2}\sqrt{\pi}.\frac{1}{3}\sqrt{\pi}}{2.2.1}+\frac{1}{2}b^2\frac{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}.\sqrt{\pi}}{2.2.1}\right\}=4ab\left\{\frac{1}{16}\pi a^2+\frac{1}{16}\pi b^2\right\}=\frac{\pi ab\left(a^2+b^2\right)}{4}$$

Example 10: Find the value of double integral $\iint_R e^{2x+3y}dxdy$, where R is the region bounded by $x=0$, $y=0$ and $x+y=1$.

Soln: From the figure it is clear that for the regionof integration R : y varies from $y=0$ to $y=1-x$ and x varies from $x=0$ to $x=1$. Hence.



(i)



(ii)

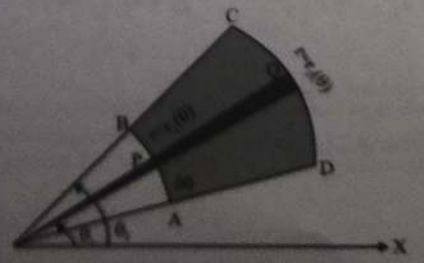
$$\iint_R e^{2x+3y}dxdy=\int_0^1\int_0^{1-x}e^{2x}e^{3y}dx\,dy=\int_0^1 e^{2x}\left\{\int_0^{1-x}e^{3y}dy\right\}dx \qquad [\text{x constant}]$$

$$=\int_0^1 e^{2x}\left[\frac{e^{3y}}{3}\right]_0^{1-x}dx=\frac{1}{3}\int e^{2x}.\left[e^{3(1-x)}-e^0\right]dx$$

$$=\frac{1}{3}\int_0^1\left(e^{3-x}-e^{2x}\right)dx=\frac{1}{3}\left[-e^{3-x}-\frac{e^{2x}}{2}\right]_0^1=\frac{1}{3}\left[\left\{-e^2-\frac{e^2}{2}\right\}-\left\{-e^3-\frac{e^0}{2}\right\}\right]$$

$$=\frac{1}{3}\left[\left\{-e^2-\frac{e^2}{2}\right\}-\left\{-e^3-\frac{e^0}{2}\right\}\right]=\frac{1}{3}\left(-\frac{3}{2}e^2+e^3+\frac{1}{2}\right)=\frac{1}{3}e^3-\frac{1}{2}e^2+\frac{1}{6}$$

Evaluation of Double Integrals in Polar Co-ordinates



We have to evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}f(r,\theta)dr\,d\theta$ over the region bounded by the straight lines $\theta=\theta_1$ and $\theta=\theta_2$ and the curve $r=r_1(\theta)$ and $r=r_2(\theta)$. We first integrate with respect to r between the limits $r=r_1(\theta)$ and $r=r_2(\theta)$ and taking θ as constant. Then the resulting expression is integrated with respect to θ between the limits $\theta=\theta_1$ and $\theta=\theta_2$.

The area of integration is ABCD. On integrating first with respect to r, the strip extends from P to Q and the integration with respect to θ means the rotation of this strip PQ from AD to BC.

Example 11: Evaluate $\int_0^\pi\int_0^{a\theta}r^3 d\theta dr$

Soln: We have, $I=\int_0^\pi\left[\frac{r^4}{4}\right]_0^{a\theta}d\theta=\frac{1}{4}\int_0^\pi a^4 d\theta=\frac{a^4}{4}\left[\frac{\theta^5}{5}\right]_0^\pi=\frac{a^4\pi^5}{20}$

Example 12: Evaluate $\int_0^\pi\int_0^{a(1+\cos\theta)}r^3\sin\theta\cos\theta d\theta dr$

Soln: We have, $I=\int_0^\pi\sin\theta\cos\theta\left[\int_0^{a(1+\cos\theta)}r^3dr\right]d\theta=\int_0^\pi\sin\theta\cos\theta\left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)}d\theta$

$$=\frac{a^4}{4}\int_0^\pi(1+\cos\theta)^4\sin\theta\cos\theta d\theta$$

Put $1+\cos\theta=t$ and $-\sin\theta.d\theta=dt$

$$=\frac{a^4}{4}\int_2^0 t^4(t-1)(+dt)=\frac{a^4}{4}\int_0^2\left(t^5-t^4\right)dt=\frac{16}{15}a^4$$

Example 13: Evaluate $\iint_A r^2\sin\theta d\theta dr$ over the area of cardioid $r=a(1+\cos\theta)$ above the initial line.

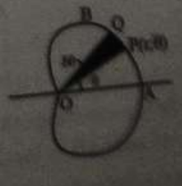
Soln: The region of integratin A can be covered by radial strips whose ends are at $r=0, r=a(1+\cos\theta)$

The strips lie between $\theta=0$ and $\theta=\pi$

Thus $\iint_A r^2\sin\theta d\theta dr=\int_0^\pi\int_0^{a(1+\cos\theta)}r^2\sin\theta d\theta dr$

$$=\int_0^\pi\sin\theta\left[\int_0^{a(1+\cos\theta)}r^2dr\right]d\theta$$

$$=\int_0^\pi\sin\theta\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}d\theta$$



$$=\frac{a^3}{3}\int_0^\pi(1+\cos\theta)^3\sin\theta d\theta=\frac{16a^3}{3}\int_0^\pi\cos^7\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$$

$$=\frac{16}{3}a^3\int_0^{\pi/2}\sin\phi\cos^7\phi.2d\phi \qquad \text{putting } \frac{\theta}{2}=\phi$$

$$=2\times\frac{16}{3}a^3\left[\frac{-\cos^8\phi}{8}\right]_0^{\pi/2}=\frac{4}{3}a^3$$

Charge of order of integration :

On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration.

Some of the problems connected with double integrals, which seem to be complicated, can be made easy to handle by a change in the order of integration.

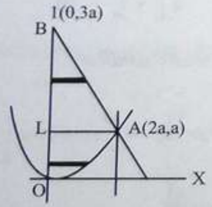
Example 14: Change the order of integration in $\int_0^{2a}\int_{x^2/4a}^{3a-x} F(x,y)\,dx\,dy$

Soln: Here the limits are $x^2 = 4ay$ and $y = 3a - x$; $x^2 = 4y$ and $x + y = 3a$ and also $x = 0$ and $x = 2a$.

The integral is extended to area OAB. We consider shaded strip.

Now, the integral is broken into two parts which are corresponding to the area OAL and the triangle BLA.

Hence we have



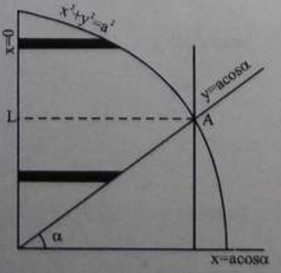
$$\int_0^{2a}\int_{x^2/4a}^{2a-x} F(x,y)\,dxdy = \int_0^{a}\int_0^{2\sqrt{ay}} F(x,y)\,dy\,dx + \int_0^{3a}\int_0^{3a-y} F(x,y)\,dydx$$

Since $x^2 = 4ay$, i.e., $x = 2\sqrt{ay}$

Example 15: Change the order of integration into the double integral.

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)\,dxdy$$

Soln: The region of integration is bounded by the line $y = x\tan\alpha$, the circle $y^2 = a^2 - x^2$, *i.e.*, $x^2 + y^2 = a^2$, the y-axis, $x = 0$ and line $x = a\cos\alpha$. Clearly, the region of integration is OAB.



The strips parallel to x-axis change their character at $A(a\cos\alpha, a\sin\alpha)$. Thus the region is divided into two parts namely OAL and LAB. In the region OAL, the strip parallel to x-axis has its extremities on $x = 0$ and $y = x\tan\alpha$. Hence limits of x are from 0 to $y\cot\alpha$. As the point A is $(a\cos\alpha, a\sin\alpha)$. Again in the region LAB, the strip parallel to x-axis has its extremities on $x = 0$ and circle $x^2 + y^2 = a^2$. Hence limits of x are from 0 to $\sqrt{(a^2-y^2)}$. The limits of y are clearly from $a\sin\theta$ to a.

Hence, we have

$$\int_0^{a\sin\theta}\int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)\,dxdy = \int_0^{a\cos\alpha}\int_0^{y\cot\alpha} f(x,y)\,dydx + \int_{a\sin\theta}^{a}\int_0^{\sqrt{(a^2-y)}} f(x,y)\,dydx$$

To verify the above result when $f(x,y) = 1$. In this case the integral becomes $\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} xdy$ and represents the area of the vector of the circle radius a, the angle of sector $BOA = \frac{1}{2}\pi - \alpha >$ Hence the result should be $\frac{1}{2}a^2\left(\frac{1}{2}\pi - \alpha\right)$. Also transformed integral is:

$$\int_0^{a\sin\alpha_0}\int_0^{y\cot\alpha} dydx + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} dydx = \int_0^{a\sin\alpha} y\cot\alpha\,dy + \int_{a\sin\alpha}^{a}\sqrt{(a^2-y^2)}\,dy$$

$$= \int_0^{a\sin\alpha} y\cot\alpha\,dy + \int_{a\sin\alpha}^{a}\sqrt{(a^2-y^2)}\,dy = \left[\frac{1}{2}y^2\cot\alpha\right]_a^{a\sin\alpha} + \left[\frac{1}{2}y\sqrt{(a^2-y^2)} + \frac{1}{2}a^2\sin^{-1}(y/a)\right]_{a\sin\alpha}^{a}$$

$$= \frac{1}{2}a^2\sin\alpha\cos\alpha + \frac{1}{2}a^2\left[\frac{1}{2}\pi - \sin\alpha\cos\alpha - a^2\alpha\right] = \frac{1}{2}a^2\left(\frac{1}{2}\pi - \alpha\right)$$

This verifies the result.

Example 16: Change the order of integration in

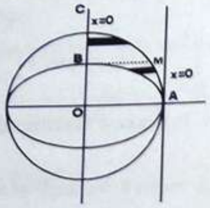
$$\int_0^{a}\int_{\frac{1}{2}\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} V\,dxdy \qquad (V \text{ being a function of } x \text{ and } y)$$

Soln: The region of integration is bounded by $y = \frac{1}{2}\sqrt{(a^2-x^2)}$ *i.e.* $x^2 + 4y^2 = a^2$ (an ellipse), the circle $x^2 + y^2 = a^2$, $x = 0$ (y-axis) and the line $x = a$, the positive quadrant. Clearly, the region is ABC. The strip parallel to x-axis change its character at B where $y = \frac{1}{2}a$.

Hence the region is divided into two parts namely ABM and BCM.

The strip parallel to x-axis, in the region ABM, has its extremities on the ellipse $x^2 + 4y^2 = a^2$ and the circle $x^2 + y^2 = a^2$. Hence limits of x are from $\sqrt{(a^2-4y^2)}$ *to* $\sqrt{(a^2-y^2)}$. Limits of y are form 0 to $\frac{1}{2}a$ B is the point $\left(0, \frac{1}{2}a\right)$.

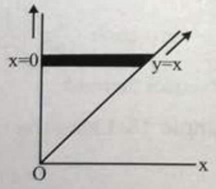
In the region BCM, a strip parallel to x-axis has its extremities on $x = 0$ and the circle $x^2 + y^2 = a^2$. Hence limits of x are from 0 to $\sqrt{(a^2-y^2)}$. Also limits of y are from $\frac{1}{2}a$ to a.



Hence, $\int_0^a \int_{\frac{1}{2}\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} V dx\, dy = \int_0^{\frac{1}{2}a} \int_{\sqrt{(a^2-4y^2)}}^{\sqrt{(a^2-y^2)}} V dxdy + \int_{\frac{1}{2}a}^{a} \int_0^{\sqrt{(a^2-y^2)}} V dydx$

Example 17: Change the order of integration in $\int_0^\infty \int_0^\infty \frac{e^{-y}}{y} dxdy$ and hence find its value.

Soln: The region of integration is bounded by lines $y = x, x = 0$, (y-axis) and an infinite boundary.
Take strips parallel to x-axis to change the order of integration.
Take strips parallel to x-axis to change the order of integration.
The extremities of the strip lies on x = 0 and y = x.
Hence, limits of x are from 0 to y and limits of y are from 0 to ∞.



$$\therefore \quad \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dydx = \int_0^\infty \frac{e^{-y}}{y} . [x]_0^y \, dy$$

$$= \int_0^\infty e^{-y} dy = \left[-e^{-y}\right]_0^\infty = 1$$

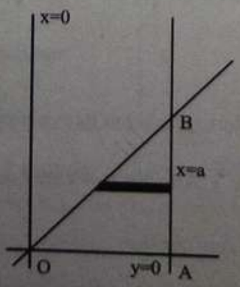
$$= \int_0^{ab/\sqrt{(a^2+b^2)}} \int_0^{ab/\sqrt{(a^2+b^2)}} f(x,y)\, dxdy + \int_{ab/\sqrt{(a^2+b^2)}}^{a} \int_0^{(a/b)\sqrt{(a^2-b^2)}} f(x,y)\, dxdy$$

Example 18: Change the order of integration in $\int_0^a \int_0^x \frac{f'(y)dy}{\sqrt{(a-x)(x-y)}}$ and hence find its value.

Soln: The region of integration is bounded by

$$y = 0, y = x, x = 0 \text{ and } x = a$$

Thus OAB is the region of integration. Consider an elementary strip parallel to the axis of x. This strip has its extremities on $y = x$ and $x = a$. Hence limits of x are from y to a.



Also y varies from O to B; so its limits are from 0 to a. Hence the integral after changing the order of integration is

$$= \int_0^a \int_y^a \frac{f'(y)\, dydx}{\sqrt{(a-x)(x-y)}}$$

To find its value, put

$$x = a\sin^2\theta + y\cos^2\theta,\ dx = 2(a-y)\sin\theta\cos\theta d\theta,$$

$$a - x = (a-y)\cos^2\theta,\ x - y = (a-y)\sin^2\theta$$

When $x = y, \sin^2\theta = 0$, i.e. $\theta = 0$ and when $x = a, \cos^2\theta = 0$ i.e. $x = 1/2\pi$

Now, $\int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}} = \int_0^{\pi/2} \frac{2(a-y)\sin\theta\cos\theta d\theta}{(a-y)\sin\theta\cos\theta} = 2\int_0^{\pi/2} d\theta = \pi$

$\therefore \quad$ Given integral $= \int_0^a \pi f'(y)\, dy = \pi\left[f(y)\right]_0^a = \pi\left[f(a) - f(0)\right]$

PRACTICE SET

Evaluate the following double integrals:

1. $\int_1^2 \int_0^{3y} y dydx$ 2. $\int_0^2 \int_0^x \frac{1}{x^2+y^2} dxdy$ 3. $\int_0^2 \int_0^{\sqrt{(1+x^2)}} \frac{1}{x^2+y^2} dxdy$

4. $\int_0^1 \int_0^{x^2} e^{y/x} \, dx\, dy$ 5. $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} x dxdt$ 6. $\int_0^1 \int_y^{\sqrt{y}} \left(x^2+y^2\right) dydx$

7. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{1}{1+x^2+y^2} dxdy$ 8. $\int_0^1 \int_0^{\pi/2} \cos(x+y)\, dydx$ 9. $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2-x^2-y^2)}\, dydx$

10. $\int_0^\pi \int_0^{a\sin\theta} r d\theta dr$ 11. $\int_0^{\pi/2} \int_0^{a\cos\theta} r\sin\theta d\theta dr$

12. $\int_0^{\pi/2} \int_0^a r^n \sin^n\theta \cos\theta d\theta dr$, for $n+1>0$

13. $\iint \frac{r d\theta dr}{\sqrt{(a^2+r^2)}}$ over one loop of $r^2 = a^2\cos 2\theta$ 14. $\iint r^2 d\theta dr$ over the area of the circle $r = a\cos\theta$

15. Evaluate $\iint x^2 y^2 dxdy$ over the region $x^2 + y^2 \le 1$.

16. Evaluate $\iint_A \left(x^2+y^2\right) dxdy$ over the region bounded by $x = 0, y = 0, x + y = 1$

17. Evaluate $\iint_R dxdy$, where R is the positive quadrant of the circle.

18. Evaluate $\iint_A \frac{xy}{\sqrt{(1-y^2)}} dxdy$, where the region of integration in the positive quadrant of the circle $x^2 + y^2 = 1$

19. Evaluate $\iint xy dxdy$ over the region in the positive quadrant for which $x + y \le 1$.

20. Evaluate $\iint_R y\,dx\,dy$ over the part R of the plane bounded by the line y = x and the parabola $y = 4x - x^2$.

21. Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

22. Evaluate $\iint_R \frac{xy}{x^2+y^2}\,dx\,dy$, where R is the region of integration bounded by $y = x$, $y = 2x$, $x = 2$.

ANSWER KEY

(1) 7

(2) $\frac{\pi}{4}\log 2$

(3) $\log\left(1+\sqrt{2}\right)$

(4) $\frac{1}{2}$

(5) $\frac{\pi}{2}$

(6) $\frac{3}{35}$

(7) $\frac{\pi}{4}\log\left(1+\sqrt{2}\right)$

(8) -2

(9) $\frac{\pi a^3}{6}$

(10) $\frac{\pi a^2}{4}$

(11) $\frac{a^2}{6}$

(12) $\frac{a^{n+1}}{(n+1)^2}$

(13) $\left(2-\frac{\pi}{2}\right)a$

(14) $\frac{\pi a^3}{9}$

(15) $\frac{\pi}{24}$

(16) $\frac{1}{6}$

(17) $\frac{1}{6}$

(18) $\frac{1}{24}$

(19) $\frac{1}{24}$

(20) $\frac{54}{8}$

(21) $\frac{48}{4}$

(22) $\log\frac{5}{2}$